Fast Approximation of the Tangent, Hyperbolic Tangent, Exponential and Logarithmic Functions

Abstract

There are some of us who enjoy using our wits and number sense to calculate mathematical quantities as they are encountered. This is surely baffling to the rest of the population, but there it is. A good discriminator is this paper, which offers novel methods for manually or mentally calculating the tangent, hyperbolic tangent, exponential and logarithmic functions to essentially four significant digits. Depending on which population group you fall into, I suspect that further enticement is either unnecessary or insufficient.

P̲URSUITS IN MATHEMATICS often produce elementary functions as solutions that need to be evaluated. This paper offers a means for fast, convenient calculation of four of these functions. The tangent function $\tan x$, though difficult to approximate well, is encountered often enough that a fast, accurate approximation can be very useful tool. The hyperbolic tangent $\tanh x$ is less common, but the exponential function e^x and the logarithmic function $\ln x$ appear quite often in technical work. All of these functions can be evaluated by the same general method, so we will cover each of them individually in this paper.

We make use here of the fact that the tangent, hyperbolic tangent, and exponential functions have addition formulas $\tan(a+b)$, $\tanh(a+b)$, and e^{a+b} that involve simple functions of the same type. This property is useful because we can iteratively construct, say, $\tan x$ with combinations of smaller, known values of $\tan a$, $\tan b$, and so forth. If the known values are simple fractions, the addition formulas are greatly simplified. We carefully choose to memorize a very small number of these that easily combine to span the required range of the functions to good accuracy. We can then quickly approximate these functions to within a small error, at which point we make use of a final small correction to obtain four-digit accuracy. The object is to create simple, fast algorithms with reasonable accuracy that can be performed initially using pencil and paper, and with practice can be performed mentally.

This paper begins with a short overview of methods of mental division, followed by sections presenting approximation techniques for these functions based on simplified addition formulas.

A Word or Two about Division...

THERE ARE INSTANCES, as in the later sections of this paper, where we are called upon to divide by, say, a four-digit value. We would like to perform these very quickly to our accuracy, and eventually we want to do these mentally. Efficient methods of mental division are rarely covered in books on mental math, but an aptitude for such things is critical for many algorithms. Let's take a minute to discuss the ways that we can approach these situations.

Our goal is to reduce the denominator to an integer of one or two digits at most, as short division by numbers of this size are not too difficult. First, we convert the denominator to an integer by shifting its decimal point and shifting the decimal point in the numerator by the same amount. For example, $4.657/.07 = 465.7/7 = 66.53$ to four digits. Then we look to simplify the fraction by dividing the numerator and denominator by low common factors. For example, $.2420/7.2 = 2.420/72 = .605/18 = .0336$ to four digits. We could have twice divided through by 2, but the last two digits of both numbers are divisible by 4, so the entire numbers are divisible by 4. The division by 18 can be done directly (I would count up by 18's here, so for 60 we have $18 \Rightarrow 36 \Rightarrow 54$ gives 3 remainder 6, then for 65 we know 54 again gives 3 remainder 11, then for 110 we double 54 to give 6 remainder 2, etc.), or we can divide .605 by 2, then by 9. Division by 2 is easiest if the number is split into even number groups, so .605 is split into .(60)(50), so half of each even group gives .3025, and dividing this by 9 yields .0336 as before. In other words, we can divide the denominator by a convenient factor even when the numerator is not evenly divisible by it, e.g., $35/36 = 5.833/6 = .9722$.

We can also adjust the denominator a little bit to get it to a round number as long as we adjust the numerator by the same percentage. If we are solving $247/119$, we see that the numerator is about twice the denominator, so if we adjust 119 up to 120, we need to adjust 247 by about 2, and we arrive at $249/120 = 24.9/12 = 2.0750$ compared to the actual value of $2.0756\ldots$. With experience, we might notice that 247 is twice 119 plus about 10%, so we could add 2.1 to 247 to get a more accurate $24.91/12 = 2.0758$. If we have $91.5/353$, we can adjust the denominator down to 350 and double the fraction to have a single-digit division, so $91.5/353 = 90.75/350 = 181.5/700 = 1.815/7 = .2593$, where we reasoned that decreasing 353 by 3 was roughly equal to decreasing 91.5 by $3/4$. Our answer will be a bit high, since 91.5 is a bit more than $1/4$ of 353, so we might subtract a tiny bit from our answer (which is in fact in excess by .0001). This shifting technique may not seem like much, but as a graduate teaching assistant I impressed more than one physics class by using it to mentally calculate answers to problems.

Finally, we can generalize an approximation that is valid for small b: $\frac{1}{1+b} \approx 1 - b$:

$$\frac{a}{c+b} \approx \frac{a}{c}\left(1 - \frac{b}{c}\right) \tag{1}$$

$$\frac{a}{c-b} \approx \frac{a}{c}\left(1 + \frac{b}{c}\right) \tag{2}$$

The error here is about .01 of a/c when b/c is $1/10$, and about .0001 of a/c when b/c is $1/100$, low for both approximations.

This is a nice alternative to shifting the denominator when the numerator is not a simple multiple or fraction of the denominator. For example, $27/61 \approx (27/60)(1 - 1/60)$. Here we can find $27/60 = 2.7/6 = .4500$, then subtract $.4500/60 = .0450/6 = .0075$ to get .4425 compared to the actual value of

.4426 . Since $^1/_{60}$ is $^1/_6$ of $^1/_{10}$ and the error follows a square law, we are low by about $(.45/36)(.01)$, or .0001 , but this is for better calculators than me.

In short, long division should not be as intimidating as it might seem, particularly since we have flexibility in our accuracy. If the problem is difficult to rearrange, we settle for less accuracy; if it can be easily manipulated, we take what we are offered.

Earlier Methods of Approximation

M Y BOOK, *Dead Reckoning: Calculating Without Instruments*,[1] derives methods for calculating the first quadrant value of $\tan d$ directly (that is, not as a ratio of the sine and cosine functions). These algorithms involve angles in units of degrees, as this occurs most commonly—a convenient conversion factor from radians to degrees is $^{401}/_7$, and of course the reciprocal of this converts degrees to radians.

These methods assume that $\tan a$ has been memorized for $a = 0°$, 10°, 20°, 30°, 40°, and 45°. The tangent algorithms cover the range 0–45°, while the range 45–90° (in which the tangent function careens to infinity) is first converted to the lower range through the identity $\tan d = 1/\tan(90° - d)$. Increasing the $\pm.0005$ (three-digit) accuracy provided would require memorizing additional values of $\tan a$, so it is desirable to find other methods of approximation that involve easier computation, greater accuracy, and few memorized values. To this end, this paper treats first the function $\tan x$ in the first quadrant, followed by the hyperbolic function $\tanh x$ (which is not treated in the book).

Methods for the exponential function and its inverse, the logarithmic function, are also given in the book. A more comprehensive summary of the exponential method can be found on my website.[2] In addition, there is an attractive alternative for computing exponentials (the McIntosh-Doerfler algorithm) that involves minimal memorization.[3] Both methods provide roughly five-digit accuracy, but they require some effort. Here we are looking for methods that are faster and easier while offering only about four-digit accuracy. The exponential function e^x and the logarithmic function $\ln x$ can benefit from the technique for the hyperbolic tangent function, so they are treated in the last sections of this paper.

Calculating the Tangent Function

T ANgent function characteristics can be leveraged to limit the range over which the calculations need to be accurate. Below is a plot of $\tan x$ for x in radians. The function is linear with x around the origin and approaches $\pm\infty$ as x approaches $\pm\pi/2$, or $\pm90°$. The plot here is truncated at $x = \pm\pi/4 = \pm.7854$ or $\pm45°$, where $\tan x = \pm1$. For our purposes, it is important to note that

- $\tan(-x) = -\tan x$ for all x
- $\tan x = x$ is valid to at least four decimal places when $|x| \leq 3.0°$ (or .053 radians)
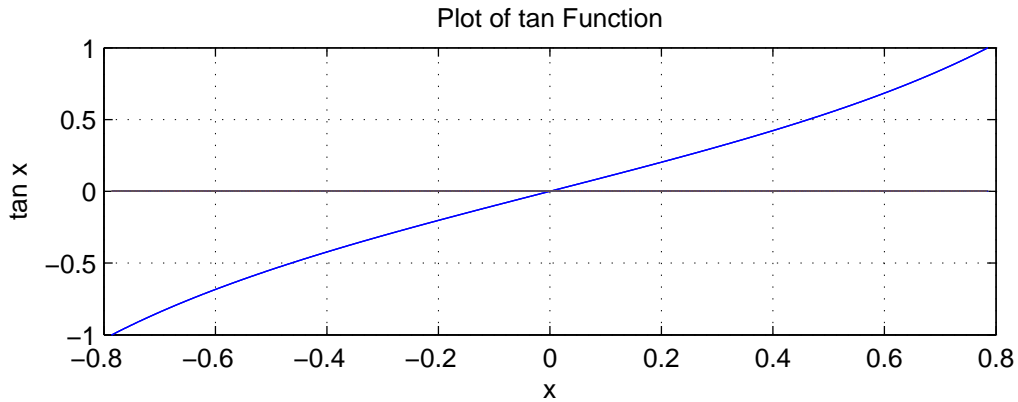
The addition formula for the tangent function is given by:

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b} \qquad (3)$$

Since $\tan(-x) = -\tan x$, the signs flip in the numerator and denominator when subtracting b. The sign in the numerator always matches the sign in the left half of the equation.

Now if $\tan a = {}^{N_1}\!/_{D_1}$ and $\tan b = {}^{N_2}\!/_{D_2}$, the addition formula simplifies to

$$\tan(a+b) = \frac{N_1 D_2 + N_2 D_1}{D_1 D_2 - N_1 N_2} \qquad (4)$$

A convenient mental picture of Equation (4) is given in the diagram below, representing the numerator as the sum of the N–D cross-products and the denominator as the negative N product plus the D product.
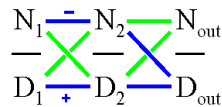
In the same way as for Equation (3), we flip the signs in the numerator and denominator of Equation (4) to subtract b.

Finally, for very small b in radians, we can derive a correction c that we add to $\tan a$ to find $\tan(a+b)$:

$$\begin{aligned}
\tan(a+b) &= \frac{\tan a + b}{1 - b \tan a} \\
&\approx (\tan a + b)(1 + b \tan a) \\
&\approx \tan a + b(1 + \tan^2 a)
\end{aligned}$$

or,

$$\boxed{c = b(1 + \tan^2 a)} \qquad (5)$$

4

where we limit the calculation to our four-digit accuracy. We will use this correction when b is within $\pm.28°$ (or $\pm.005$) of $\tan a$, as it is valid to four decimal places for even our maximum $\tan a$ of 1.

Consider the following table of some angles whose tangents are very simple fractions:

a (degrees)	a (radians)	tan a
26.57	.4636	$1/2$
18.43	.3218	$1/3$
14.04	.2450	$1/4$
11.31	.1974	$1/5$

Table 1. Angles with Simple Tangents.

There are 20 possible singles or pairs of these four angles, including pairings with themselves, and there is at least one angle or one sum or difference of a pair of angles that lies within $2.2°$ of any angle from 0–45° (or 0–.7854). In fact, comparing the two-sided $4.4°$ spacing with an average spacing of $45/19 = 2.37$ over this range, the selections are remarkably uniform in their coverage. The figure below shows the possible values. Here the numbers at the bends in the top half represent the addition of the degrees at the endpoints, the numbers in the bottom half represent differences, and the vertical arrows represent an addition of an angle to itself. The results are shown simply to demonstrate the distribution; there is no need to memorize the pairings.
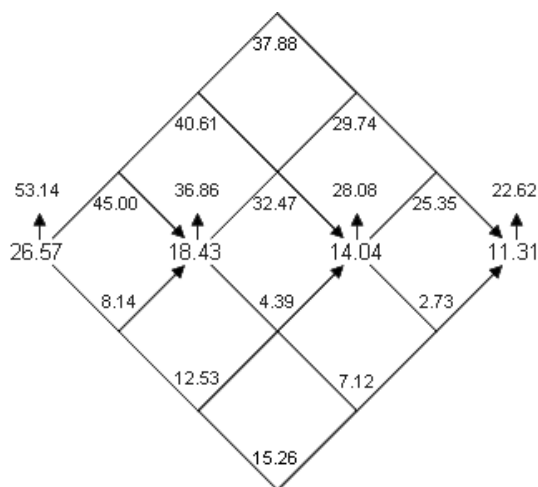
The first step of our strategy of approximating $\tan x$, then, is to identify the angle or pairwise combination of angles that provide a value x_0 that is nearest x. For a single angle, our first rough estimate is the corresponding fraction; for a pair, we use Equation (4) to add or subtract the corresponding fractions. If $(x - x_0)$ exceeds $\pm.28°$ (or $\pm.005$), we choose a fraction x_1 that is simple but approximates $(x - x_0)$ in radians. Since $(x - x_0)$ is within the linear region of the tangent function, $\tan x_1 = x_1$ to our accuracy. Therefore, we add this new fraction using Equation (4), yielding a much better approximation. Equation (5) can be used to create a final small correction, where $\tan a$ is the current approximation and $b = (x - x_0 - x_1)$. Our final answer will be accurate to nearly four

decimal places, with the error in the last digit due to rounding operations on the fifth decimal place. Of course, we can stop at any intermediate point for an answer of less accuracy.

There are two other matters that need to be addressed before we arrive at the final algorithm. The first of these is handling $\tan x$ when $x > 45°$. The second is the conversion to radians if needed, which can be simplified by carefully choosing the final fraction.

The detailed steps for approximating $\tan x$ are:

1. If $x > 45°$ (or .7854), replace x with $(90° - x)$ or $(1.571 - x)$.
2. Find the closest angle or the sum or difference of a pair of angles in Table 1 that equal an angle x_0 nearest to x. If a pair is chosen, use Equation (4) to add or subtract their fractions; otherwise the single angle fraction is our first approximation. If $(x - x_0)$ lies within $\pm.28°$ (or .005), skip to Step 4—believe it or not, this will occur almost 25% of the time.
3. Find a simple fraction x_1 roughly equal to the remaining difference $(x - x_0)$. If we are working in degrees, convert x_1 to radians. We normally do this by multiplying by $7/401$, but the less accurate $7/400$ is sufficient for our purposes here. (In this case, it is best to find an x_1 that has 4 in the numerator or 7 in the denominator so that cancellation can occur during conversion.) Then use Equation (4) to add x_1 in radians to the result of Step 2.
4. Flip the resulting fraction from Step 3 if x was replaced in Step 1. Perform the division to obtain a value to four decimal places.
5. For greater accuracy, convert the remaining angle difference $b = (x - x_0 - x_1)$ to radians if we have been working in degrees. Then calculate $c = b(1 + \tan^2 a)$ to the fourth decimal place (involving two significant digits at most), where $\tan a$ is our current estimate. If x was not replaced in Step 1, add c to our current estimate to arrive at the final approximation. If x was replaced in Step 1, subtract c instead.

Let's demonstrate this with some examples.

Example: Find $\tan 28°$.
We choose $14.04° + 14.04° = 28.08°$. From Equation (4),

$$\tan 28.08° = \frac{1(4) + 1(4)}{4(4) - 1(1)} = \frac{8}{15}$$

so $8/15$ is our first estimate.
The difference of $-.08°$ is well within the $\pm.28°$ for which Equation (5) is valid, so we skip Step 3 and proceed to the final correction of Step 4. We divide $8/15$ to get .5333 (whose square is about .25) and find $b = -.08 \times 7/400 = -.0014$.

$$c = -.0014(1 + .25) = -.0018$$
$$\tan 28° \approx .5333 - .0018 = .5315$$

which is .0002 off from the actual value of .5317.

6

If we had converted $28°$ to radians at the start of this last example, or if the original problem were given in radians, we would not have had to multiply b by $7/400$. In any event, this example did not demonstrate the entire algorithm because we skipped the final fraction. Let's choose a value that lies right between two numbers in the 20 combinations of memorized angles.

Example: Find tan .1633 .

We choose $.4636 - .3218 = .1418$, equivalent to our addition of $1/2$ and $-1/3$. Here we are subtracting $N_2/D_2 = 1/3$, so we simply flip the signs in Equation (4):

$$\tan .1418 = \frac{1(3) - (1)(2)}{2(3) + (1)(1)} = \frac{1}{7}$$

The angle difference of .0215 lies well outside the $\pm.28°$ range, so we proceed to Step 3 and find a simple fraction that approximately equals it, say $1/50 = .0200$.

$$\frac{1(50) + 1(7)}{7(50) - 1(1)} = \frac{57}{349}$$

We need to find a decimal value for this. If we increase 349 by 1 to 350, we need to increase the numerator, which is roughly $1/6$ of the denominator, by the same percentage, or by .17 instead of 1. So we instead do the division $57.17/350 = 114.34/700 = 1.1434/7 = .1633$. (Here we find our estimate equals the original radian angle, but we know it must be a little higher because we are outside the linear region).

For the greatest accuracy, we add the quantity $.0015(1 + .03)$, where .0015 is the final angle difference and .03 is a one-digit approximation of the square of our current estimate .1633 .

$$c = .0015(1 + .03) = .0015$$
$$\tan .1633° \approx .1633 + .0015 = .1648$$

which equals the actual value of .1648 to four decimal places.

Example: Find tan $23.9°$.

Again, this angle is not a fortuitous one. We can form something close to 23.9 by adding 14.04 and 11.31, or by doubling 11.31 . Choosing the first pair, we arrive at the first estimate:

$$\tan 25.35° = \frac{1(5) + 1(4)}{4(5) - 1(1)} = \frac{9}{19}$$

The angle difference $-1.45°$ lies well outside the $\pm.28°$ range, so we find a simple fraction that approximately equals it and has 4 in the numerator or 7 in the denominator, say $10/7 = 1.43$. We convert $10/7$ to radians by multiplying by $7/400$, canceling the 7's to get $1/40$. We add this to the previous estimate:

$$\frac{9(40) - 1(19)}{19(40) + 9(1)} = \frac{341}{769}$$

We need to find a decimal value for this. Again, if we increase 769 by 1 to 770, we need to increase the numerator, which is roughly half of the denominator, by the same percentage, or by .5 instead of 1. So we instead do the division $341.5/770 = 34.15/77 = .4435$.

7

Now we still have a difference of $-.02 \times {}^7/_{400} \approx -.0004$ radians, so we make our final correction for greatest accuracy. The square of .4435 is approximately .2 , so

$$c = -.0004(1+.2) = -.0005$$
$$\tan 23.9° \approx .4435 - .0005 = .4430$$

which is .0001 off from the actual value of .4431... Again, this is about the most difficult case we can encounter.

Example: Find $\tan 66.1°$.
We replace $\tan 66.1°$ with $\tan 23.9°$. We found in the previous example that $\tan 23.9° \approx {}^{341}/_{769}$, so we find the reciprocal ${}^{769}/_{341}$. Adjusting 341 down to 340 should adjust 769 down by about 2.3, so the fraction reduces to ${}^{76.67}/_{34.0}$ or 2.2550 . Subtracting the final correction $c = -.0004(1+2.25^2) = -.0024$ provides the final estimate of $2.2550 + .0024 = 2.2574$, compared to the actual value of 2.2566... We suffered a bit here because $b = -.00035$, actually, and we had rounded it to -.0004.

It might seem that we can evaluate tangents above 45° by including $\tan 45° = {}^1/_1$ and $\tan 33.69° = {}^2/_3$ in Table 1. With these additions, it turns out that all angles between 0° and 81.7° are within 3° of a single angle or the sum or difference of a pair of angles, except for a sliver between 74.6° and 75.7°. However, because of the simple relationships between the tangent values in the table, the new values are duplicates of existing ones or 90° minus existing ones, providing no advantage in the end.

In short, we have an algorithm for $\tan x$ that

- requires memorization of simple fraction tangents of only four angles (in degrees or radians or both),
- easily handles the case where $45° < x < 90°$,
- provides intermediate estimates when less accuracy is required,
- involves simple calculations with the exception of one multi-digit division for greatest accuracy,
- and provides nearly four decimal place accuracy, which is an order of magnitude better than the algorithm in my book.
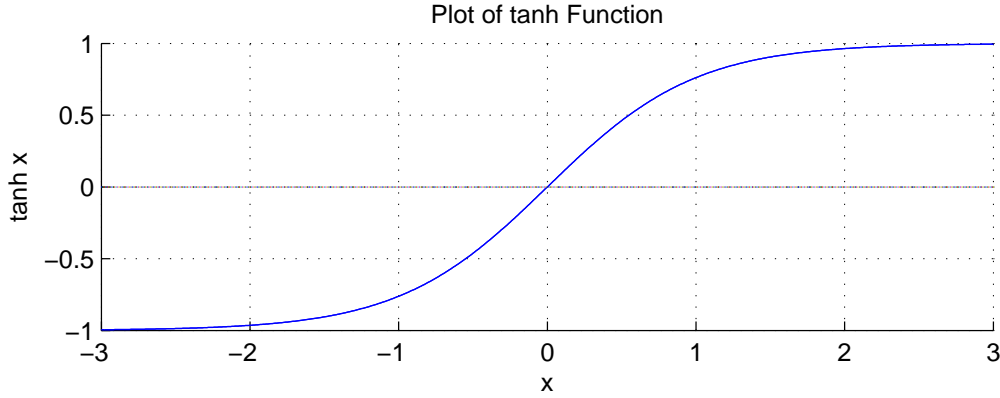
Calculating the Hyperbolic Tangent Function

TANH X, the hyperbolic tangent function, appears as the solution of a common second-order differential equation and can be defined in terms of real exponentials:

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

A plot of $\tanh x$ is given below. The function is linear with x around the origin and asymptotically approaches -1 and 1 at the extremes of x. There are no units of degrees for this function. Again, it is important for our purposes to note that

- $\tanh(-x) = -\tanh x$ for all x

- $\tanh x = x$ is valid to at least four decimal places when $|x| \leq .053$

- $|\tanh x| = 1$ to at least four decimal places when $|x| \geq 5.30$, so this is the maximum x we require in our algorithm (which by coincidence makes the range an easily remembered one)

Plot of tanh Function

The addition formula for the hyperbolic tangent function is similar to that for the tangent function, but the signs agree in the numerator and denominator:

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \cdot \tanh b} \tag{6}$$

Since $\tanh(-x) = -\tanh x$, the signs in the numerator and denominator are negative when subtracting b. The signs in the numerator and denominator always match the sign in the left half of the equation.

For $\tanh a = N_1/D_1$ and $\tanh b = N_2/D_2$, the hyperbolic tangent addition formula simplifies to the formula for the tangent function but with all signs agreeing with the sign in the left side of the equation:

$$\tanh(a+b) = \frac{N_1 D_2 + N_2 D_1}{D_1 D_2 + N_1 N_2} \tag{7}$$

Also, for very small b, we can derive a correction c that we add to $\tanh a$ to find $\tanh(a+b)$:

$$\tanh(a+b) = \frac{\tanh a + b}{1 + b \tanh a}$$
$$\approx (\tanh a + b)(1 - b \tanh a)$$
$$\approx \tanh a + b(1 - \tanh^2 a)$$

or,

$$\boxed{c = b(1 - \tanh^2 a)} \tag{8}$$

where again we limit the calculation to our four-digit accuracy. Note the sign change in the last term compared to the corresponding approximation for the tangent function, Equation (5). We can use this approximation when b is within $\pm.011$ of $\tanh a$, as then it is always valid to four decimal places.

9

Now we used an identity to reduce the domain of the tangent function to $0° \leq x \leq 45°$. There is no corresponding identity for the hyperbolic tangent function, so we need to cover the range $0 \leq x \leq 5.30$. We can use Equation (7) to add or subtract fractional values of $\tanh a$ and $\tanh b$. However, without memorizing a large number of such values, we cannot approximate $\tanh x$ to within the linear region ($\pm.053$) over our entire range of x using at most one addition or subtraction. Rather than evaluating Equation (7) multiple times, we introduce a transformation that simplifies the addition process:

$$f_a = \frac{1 + \tanh a}{1 - \tanh a} \tag{9}$$

$$\tanh a = \frac{f_a - 1}{f_a + 1} \tag{10}$$

The value f, called the *velocity factor*, is presented in a paper by Wilson[4] for simplifying relativistic addition of velocities that are fractions of the speed of light c, as the associated Lorentz transformation is identical in form to Equation (6). It turns out that if we multiply f_a and f_b, the resulting f_{ab} is the velocity factor for $\tanh(a+b)$, as shown below:

$$
\begin{aligned}
f_a \times f_b &= \frac{1 + \tanh a}{1 - \tanh a} \times \frac{1 + \tanh b}{1 - \tanh b} \\
&= \frac{1 + \tanh a \cdot \tanh b + \tanh a + \tanh b}{1 + \tanh a \cdot \tanh b - \tanh a - \tanh b} \\
&= \frac{\frac{1 + \tanh a \cdot \tanh b + (\tanh a + \tanh b)}{1 + \tanh a \cdot \tanh b}}{\frac{1 + \tanh a \cdot \tanh b - (\tanh a + \tanh b)}{1 + \tanh a \cdot \tanh b}} \\
&= \frac{1 + \frac{\tanh a + \tanh b}{1 + \tanh a \cdot \tanh b}}{1 - \frac{\tanh a + \tanh b}{1 + \tanh a \cdot \tanh b}} \\
&= \frac{1 + \tanh(a+b)}{1 - \tanh(a+b)} \\
&= f_{ab}
\end{aligned}
$$

So it is possible to find, say, $\tanh(a+b+c)$ by finding f_a, f_b and f_c using Equation (9), multiplying them to get f_{abc}, and then converting the result back to $\tanh(a+b+c)$ using the inverse Equation (10).

We can simplify this process in two ways. First, we can rewrite the transformation equations for fractional values $\tanh a = {}^N\!/_D$ and $f_a = {}^n\!/_d$:

$$f_a = \frac{D + N}{D - N} \tag{11}$$

$$\tanh a = \frac{n - d}{n + d} \tag{12}$$

The velocity factor for a negative $\tanh a$ turns out to be the reciprocal of that for a positive $\tanh a$, so when we are finding $\tanh(a-b)$ we use the reciprocal of the velocity factor for $\tanh b$.

Second and most importantly, we don't memorize $\tanh a$ fractional values and transform them to velocity factors, but rather we directly memorize values of f_a that are simple fractions. A convenient set of values are presented in the table below:

a	f_a
1.1513	10
.5493	3
.2554	$5/3$
.2027	$3/2$
.0244	$21/20$

Table 2. Angles with Simple Velocity Factors.

This table includes more memorized values than I would prefer. However, combinations of the top four values of a can approximate a given value $\tanh x$ for $0 \leq x \leq 5.30$ to roughly within the $\pm.053$ linear range. (Actually, we really should add the useful round $a = 1.7006$ for $f_a = 30$ that occurs when the top two values of f_a are multiplied and the top two values of a are correspondingly added.) Values of f_a that contain common factors were chosen so that cancellation can occur as they are multiplied (when adding a) or divided (when subtracting a). Now we could choose a simple fraction for the final remainder (as we did for the tangent function) and convert it to a velocity factor to multiply as well. However, converting the fraction to a velocity factor would complicate it, so it is preferable instead to include the last row of the table, which if needed can reduce the remainder to within nearly the $\pm.011$ range for which approximation (8) is valid to four decimal places.

The general procedure to find $\tanh x$ is to identify values of a in Table 2 that combine to approximate x. The velocity factors for added a's are multiplied and those for subtracted a's are divided. The resulting velocity factor is transformed by Equation (12) to form our first approximation. The correction c of Equation (8) is then added to arrive at the final approximation.

As usual, the calculation is simpler in practice than it probably seems from the detailed derivation. Let's work through some examples.

> ***Example:*** Find $\tanh .8121$.
> We choose $.5493 + .2554 = .8047$, so
>
> $$f = 3(5/3) = 5$$
> $$\tanh .8047 = \frac{5-1}{5+1} = 2/3 = .6667$$
>
> Now the final remainder b is .0074, and
>
> $$c = .0074(1 - (2/3)^2) = .0074(5/9) = .0041$$
> $$\tanh .8121 = .6667 + .0041 = .6708$$
>
> compared to the actual value of $.6707\ldots$

> ***Example:*** Find $\tanh 2.0324$.
> We choose $1.7006 + .5493 - .2027 = 2.0472$, so
>
> $$f = 30(3)(2/3) = 60$$
> $$\tanh 2.0472 = \frac{60-1}{60+1} = 59/61 = .9669$$

11

We approximated $59/61$ by $59/60\,(1 - 1/60)$ as described in the earlier section on division. Actually, a less accurate but faster solution for this special case is $59/61 = (1 - 1/60)/(1 + 1/60) \approx (1 - 1/60)^2 \approx 1 - 1/30 = .9667$, where we kept the only the first two terms $a^2 - 2ab$ of the binomial expansion of $(a - b)^2$.

Now the final remainder b is $-.0148$, and

$$c = -.0148(1 - .97^2) = -.0148(1 - .94) = -.0009$$
$$\tanh 2.0324 = .9669 - .0009 = .9660$$

compared to the actual value of $.9662\ldots$. Notice that b actually lies outside the .011 range for four-digit accuracy of Equation (8). We could have included .0244 as we do in the next example.

Example: Find $\tanh .7238$.
We choose $.5493 + .2027 - .0244 = .7276$, so

$$f = 3(3/2)(20/21) = 30/7$$
$$\tanh .7276 = \frac{30 - 7}{30 + 7} = 23/37 = .6216$$

To calculate the fraction $23/37$ above, we can multiply the numerator and denominator by 3 to obtain the easier division $69/111$, and in fact we can also multiply this new value $69/111$ by $9/9$ to arrive at $621/999$, producing the repeating decimal $.\overline{621}$ immediately. Now the final remainder b is $-.0038$, and

$$c = -.0038(1 - .6^2) = -.0038(.64) = -.0024$$
$$\tanh .7238 = .6216 - .0024 = .6192$$

The actual value is .6193 to four decimal places.

Is there an equivalent velocity factor for the tangent function? The tangent addition formula in Equation (3) shows that the signs in the numerator and denominator are dissimilar, unlike that in Equation (6) for the hyperbolic tangent function. We are clued to look at complex formulas since $\tanh ix = i \tan x$, or $\tanh x = -i \tan ix$, and poking around we find for the tangent function,

$$g_a = \frac{1 + i \tan a}{1 - i \tan a}$$
$$\tan a = i\left(\frac{1 - g_a}{1 + g_a}\right)$$

This works, but velocity factors provide no overall benefit for the tangent function due to the complication of multiplying complex numbers and the fact that only one pair of a values are needed at most in the tangent calculation.

We now have an algorithm for the hyperbolic tangent function that provides nearly four decimal place accuracy over the entire range of the function, providing an intermediate approximation and a correction for greater accuracy.

Calculating the Exponential Function

eXponentials can also benefit from our work so far. Watch as we do some algebraic manipulations on the tanh x definition given earlier:

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$= 1 - \frac{2}{e^{2x} + 1}$$

$$\frac{2}{e^{2x} + 1} = 1 - \tanh x$$

$$\frac{2}{1 - \tanh x} = e^{2x} + 1$$

$$e^{2x} = \frac{2}{1 - \tanh x} - 1$$

$$= \frac{2 - (1 - \tanh x)}{1 - \tanh x}$$

$$= \frac{1 + \tanh x}{1 - \tanh x}$$

or,

$$e^{2x} = f_x$$

where f_x is the velocity factor once again!

This is quite surprising, and it means that we can approximate e^{2x} by multiplying those velocity factors f_a from Table 2 whose values of a combine to approximate x. Furthermore, for a final remainder of b, we can approximate e^b to four digits in the same range guaranteed by the tanh a table ($0 \le b \le .012$) with

$$e^b \approx 1 + 1.01b \tag{13}$$

Since we multiply our net velocity factor f_{net} by e^{2b} for our final value, we effectively add the following correction c to f_{net}:

$$\boxed{c = 2(1.01)bf_{net}} \tag{14}$$

To find e^{2x}, we first divide the exponent by 2 to find x. Then we adjust x by multiples of ± 1.1513 to lie within the range $0 \le x \le 1.1513$, which corresponds to the value $f_a = 10$ in Table 2. From that table we find f_{net} by multiplying f_a values for combinations of a values that approximate x (remember to use the reciprocal of f_a when subtracting a). For the final remainder b we add the value c from Equation (14). Then since $e^{2(1.1513)} = 10$, we multiply our last result by 10 for each multiple we subtracted, or divide by 10 for each multiple we added, to arrive at e^{2x} to roughly four-digit accuracy.

Because of its symmetry about the origin, the tangent of a negative number can simply be calculated as the negative of the tangent of the absolute value of the number; this is true for the hyperbolic tangent as well. However, the exponential function has no such symmetry, so e^{-x} has to be evaluated directly. We could add the multiples described above to bring x to a positive number and adjust the final result. Or, we can start with the reciprocal of a velocity factor to provide an initial negative value of a. This is demonstrated in the second and fourth examples below.

Example: Find $e^{1.5246}$.
We divide by 2 to find $x = .7623$. We choose $.5493 + .2027 = .7520$, so we have remainder $b = .0103$ and

$$f = 3\,(3/2) = 9/2$$
$$c = 2(1.01)(.0103)\,(9/2) = 1.01(.0927) = .0927 + .0009 = .0936$$
$$e^{1.5246} = 9/2 + .0936 = 4.5936$$

compared to the actual value of $4.5933\ldots$

Example: Find $e^{-.6850}$.
We divide by 2 to find $x = -.3425$. We choose $-.5493 + .2027 = -.3466$, so we have remainder $b = .0041$ and

$$f = (1/3)\,(3/2) = 1/2$$
$$c = 2(1.01)(.0041)\,(1/2) = .0041 + .0000 = .0041$$
$$e^{-.6850} = 1/2 + .0041 = .5041$$

which agrees with the actual value to four decimal places.

Example: Find $e^{6.2037}$.
We divide by 2 to find 3.1019 and subtract $2(1.1513) = 2.3026$ to arrive at $x = .7993$. We choose $.5493 + .2554 = .8047$, so we have remainder $b = -.0054$ and

$$f = 3\,(5/3) = 5$$
$$c = 2(1.01)(-.0054)(5) = 1.01(-.0540) = -(.0540 + .0005) = -.0545$$
$$e^{6.2037} = 10^2(5 - .0545) = 494.55$$

compared to the actual value of 494.58.

Example: Find $e^{-6.36}$.
We divide by 2 to find -3.18 and add $2(1.1513) = 2.3026$ to arrive at $x = -.8774$. We choose $-1.1513 + .2554 + .0244 = -.8715$, so we have remainder $b = -.0059$ and

$$f = (1/10)(5/3)(21/20) = 7/40$$
$$c = 2(1.01)(-.0059)(7/40) = 1.01(-.0826/40) = -.0834/40$$
$$e^{-6.36} = (7 - .0834)/4000 = .001729$$

which is the actual value to this four-digit accuracy. This was hard! It is about the most difficult problem of this type we can encounter.

Now $10^x = e^{2.3026x}$, so we are also able to solve 10^x if we can manage the multiplication, which can be simplified to 2.303 to our accuracy. Since the first thing we do is divide the exponent by 2, though, we can eliminate that step and multiply the exponent by 1.1515 instead.

Calculating the Logarithmic Function

Logarithms can be calculated by inverting our method for e^x. The natural logarithm to base e, denoted as $\ln x$, is the most straightforward one to find by inversion. First we shift the decimal point in x left or right so that $1 < x < 10$. Then we find velocity factors in Table 2 that multiply or divide to a value x_0 that approximates x, leaving a remainder b. Our first approximation is the corresponding sums or differences of a values, multiplied by 2. Solving Equation (13) for b and doing some algebra provides the correction c:

$$c = \frac{b}{x_0}(1 - .01) \qquad (15)$$

Finally, we add 2.3026 for every decimal place we shifted left in the original x, or subtract 2.3026 for every decimal place we shifted right.

Example: Find $\ln 14.86$.
We could shift the decimal point one place to the left, but the original value is very convenient for us, as it is nearly $10(3/2) = 15$. Therefore, we double $1.1513 + .2027 = 1.3540$ to arrive at our first approximation, 2.7080. Our remainder $b = -.14$, so our correction is

$$c = {}^{-.14}\!/_{15}\,(1 - .01) = -.0093(1 - .01) = -.0093 + .0001 = -.0092$$
$$\ln 14.86 = 2.7080 - .0092 = 2.6988$$

compared to the actual value of 2.6987.
We could also have shifted the decimal point left to set $x = 1.486$. We can use the single velocity factor $3/2$ to approximate this, providing an initial approximation to the adjusted x of $2(.2027) = .4054$ and a remainder $b = -.014$. Then

$$c = {}^{-.014}\!/_{(3/2)}\,(1 - .01) = -.0093(1 - .01) = -.0092$$
$$\ln 14.86 = 2.3026 + .4054 - .0092 = 2.6988$$

Example: Find $\ln 348.62$.
We shift the decimal point left twice to set $x = 3.4862$. This is about $10/3 = 3.3333$. Therefore, we double $1.1513 - .5493 = .6020$ to arrive at our first approximation, 1.2040. Our remainder $b = .1529$, so our correction is

$$c = {}^{.1529}\!/_{(10/3)}\,(1 - .01) = .0459(1 - .01) = .0454$$
$$\ln 348.62 = 2(2.3026) + 1.2040 + .0454 = 5.8546$$

compared to the actual value of 5.8540.

The common logarithm $\log x$, or logarithm to base 10, can be converted to the natural logarithm by multiplying by $.4343$, which is equivalent to dividing by 2.3026. In short, Table 2 and the correction formulas provide a means for reasonably fast approximations to the exponential and logarithmic functions to nearly four digit accuracy.

Conclusion

TECHNIQUES HAVE BEEN PRESENTED in this paper for approximating the tangent, hyperbolic tangent, exponential and logarithmic functions to nearly four-digit accuracy. The addition formulas for these functions consist of simple relationships between the same functions, so it is possible here to add or subtract a set of known values to approximate an unknown value. This is not possible, for example, for the sine and cosine functions, as $\sin(a+b)$ and $\cos(a+b)$ in their simplest form mix sine and cosine terms. The memorization and arithmetic operations in the methods of this paper are designed to minimize the effort involved; they include the two tables, the final corrections c, and the algorithms themselves. These approximations are intended for quick pencil-and-paper calculations or, with practice, fast mental solutions. They form a convenient set of quite accurate calculational tools.

References

[1] Doerfler, R., *Dead Reckoning: Calculating Without Instruments*, Gulf Publishing Company: Houston, 1993.

[2] Doerfler, R., *The Practical Use of the Bemer Method for Exponentials*.
http://www.myreckonings.com/Dead_Reckoning/Chapter_4/Materials/Bemer_Exponentials.pdf

[3] McIntosh, J., *Exponentials*.
http://www.urticator.net/essay/6/641.html

[4] Wilson, Alma Teao, *Using Ordinary Multiplication To Do Relativistic Velocity Addition*, arXiv:physics/0611192v1, 21 Nov., 2006.
http://arxiv.org/PS_cache/physics/pdf/0611/0611192.pdf