General Method for Extracting Roots using (Folded) Continued Fractions by Manny Sardina

1. INTRODUCTION

I'd seen methods before that extract roots in various ways, but wanted to find a way of getting a rapid result, with the least effort and with a relatively good degree of accuracy. This initial idea set me on an unexpectedly short journey, the results of which are in the following text.

I have kept the various methods and their derivation as simple as possible. There are few variables used which can be summarised in the following general formula.

$$r = (y)^{m/n} = (\alpha^n + \beta)^{m/n} = \alpha^m + \delta$$

r is the root of the initial number y. n is the root, for example, $(y)^{1/3}$ is the cube root of y. m is the "power". Throughout I will group m/n under the term root. α is the initial estimate of the root of $(y)^{1/n}$. β is equivalent to $y - \alpha^n$. The last variable is δ, which is the part we will be focusing on most of all, and is the correcting value to α^m to give ever closer approximations to the root of $(y)^{m/n}$.

Applying values to the variables above for $(65)^{2/3}$ would give

$$r = (65)^{2/3} = (4^3 + 1)^{2/3} = 4^2 + \delta$$

The derivations that follow are only for positive values of y and I have in no way attempted to explore roots of negative numbers.

2. ROOTS OF THE FORM $(y)^{1/n}$

This section deals with roots such as square roots and cube roots, closing with a general formula for any positive integer value of n.

2.1 Square Roots by Example $\sqrt{65}$

The actual answer is 8.062257748…

We begin by taking the initial formula given in the introduction.

2.1a $r = (y)^{m/n} = (\alpha^n + \beta)^{m/n} = \alpha^m + \delta$

The square root of 65 will lie between 8 and 9. 8^2 being 64 and 9^2 being 81. Assigning the appropriate values we have $y = 65, m = 1, n = 2, \alpha = 8, \beta = 1$. δ will lie in the range of 0 to 1. Substituting these values into 2.1a gives

2.1b $r = (65)^{1/2} = (8^2 + 1)^{1/2} = 8 + \delta$

The aim here is to create a formula that gives a value for δ. Working with the last two parts of 2.1b gives

2.1c $(8^2 + 1)^{1/2} = 8 + \delta$

2.1d $8^2 + 1 = (8 + \delta)^2$

2.1e $64 + 1 = 64 + 16\delta + \delta^2$, cancelling the 64s and rearranging gives

2.1f $1 = \delta(16 + \delta)$, and then

2.1g $\delta = \dfrac{1}{16 + \delta}$

This formula will be used iteratively to calculate closer approximations to the true value of δ and is therefore rewritten as

2.1h $\delta_1 = \dfrac{1}{16 + \delta_0}$

δ_0 is relatively small and for the second approximation, the first being our initial estimate for α of 8, will be

$r_2 = \alpha + \delta_1 = 8 + \dfrac{1}{16} = 8.0625$ which is correct to 3 decimal places.

The next approximation of δ is achieved by substituting the calculated value of δ_1 into 2.1h in place of δ_0. Which gives

2.1i $\delta_2 = \dfrac{1}{16 + \dfrac{1}{16 + \delta_0}}$

δ_0 is again ignored, because of its relative size, giving

2.1j $$\delta_2 = \frac{1}{16 + \frac{1}{16}} = \frac{16}{257} = 0.062256809...$$

The third approximation of the root is therefore

$r_3 = \alpha + \delta_2 = 8 + 0.062256809 = 8.062256809$, correct to 5 decimal places.

The next approximation of δ, δ_3 is achieved in the same way by substituting the value of δ_2 in 2.1i into 2.1h replacing δ_0. This gives

2.1k $$\delta_3 = \frac{1}{16 + \frac{1}{16 + \frac{1}{16 + \delta_0}}}$$

δ_0 has served its purpose, and is again ignored giving

2.1l $$\delta_3 = \frac{1}{16 + \frac{1}{16 + \frac{1}{16}}} = \frac{1}{16 + \frac{16}{257}} = \frac{257}{4128} = 0.062257751...$$

The fourth approximation of the root is then

$r_4 = \alpha + \delta_3 = 8 + 0.062257751 = 8.062257751$, correct to 7 decimal places.

This can be continued to any degree, and will see a gain of approximately 2 decimal places for each additional iteration. The gain of 2 decimal places is only applicable to this specific calculation. Other numbers will converge at different rates. This will be explored in a later section.

This is only the beginning. In later sections the rate of convergence is increased and for this particular calculation, $\sqrt{65}$, there will be a net gain of 4 decimal places for each additional step.

2.2 Linear Expansion of Continued Fractions

For the mental calculator it can be advantageous to have a fractional estimate but the process of transforming the fraction to a decimal can be an arduous task. This section provides a possible solution to that particular issue.

The method is to first of all transform the resultant continued fraction into one where the numerators in each case are 1, and then to transform this into standard fractions which can be summed in a linear way using estimates of their true value. First I'll derive a general formula and then use this to transform the fraction in 2.1j to a decimal approximation.

Beginning with the simple case of

2.2a $\frac{1}{A+\frac{1}{B}}$, which can be transformed to

2.2b $\frac{B}{AB+1}$

The initial estimate would have been $\frac{1}{A}$ so the new fraction will be $\frac{B}{AB+1}-\frac{1}{A}$ which is

2.2c $\frac{AB-AB-1}{A(AB+1)}=\frac{-1}{A(AB+1)}$

The linear expansion of $\frac{1}{A+\frac{1}{B}}$ is therefore

2.2d $\frac{1}{A+\frac{1}{B}}=\frac{1}{A}-\frac{1}{A(AB+1)}$

Adding an additional fraction $\frac{1}{C}$

2.2e $\frac{1}{A+\frac{1}{B+\frac{1}{C}}}=\frac{1}{A+\frac{C}{BC+1}}=\frac{BC+1}{ABC+A+C}$

Subtracting the first two fractions in 2.2d results in the third linear fraction

2.2f $\frac{1}{(AB+1)(ABC+A+C)}$

2.2g $$\frac{1}{A+\frac{1}{B+\frac{1}{C}}}=\frac{1}{A}-\frac{1}{A(AB+1)}+\frac{1}{(AB+1)(ABC+A+C)}$$

Taken to the fourth fraction gives

2.2h

$$\frac{1}{A+\frac{1}{B+\frac{1}{C+\frac{1}{D}}}}=\frac{1}{A}-\frac{1}{A(AB+1)}+\frac{1}{(AB+1)(ABC+A+C)}-\frac{1}{(ABC+A+C)(ABCD+AD+CD+1)}$$

Returning to the example in 2.1j gives

2.2i $$\frac{1}{16+\frac{1}{16}}=\frac{1}{16}-\frac{1}{16(16*16+1)}=\frac{1}{16}-\frac{1}{16(257)}$$

$\frac{1}{16(257)}$ is $\frac{1}{4*4*257}=\frac{1}{4*1028}=\frac{1}{4112}\approx\frac{1}{4000}$

$\frac{1}{16}$ is 0.0625 and $\frac{1}{4000}$ is 0.00025

$0.0625-0.00025=0.06225$ which is good to 5 decimal places.

Another example, using the fraction 9/43 would develop as follows

$$\frac{9}{43}=\frac{1}{\frac{43}{9}}=\frac{1}{4+\frac{7}{9}}=\frac{1}{4+\frac{1}{\frac{9}{7}}}=\frac{1}{4+\frac{1}{1+\frac{2}{7}}}=\frac{1}{4+\frac{1}{1+\frac{1}{\frac{7}{2}}}}=\frac{1}{4+\frac{1}{1+\frac{1}{3+\frac{1}{2}}}}$$

Substituting the results of this into 2.2h gives

$$\frac{1}{4+\frac{1}{1+\frac{1}{3+\frac{1}{2}}}}=\frac{1}{4}-\frac{1}{4(4*1+1)}+\frac{1}{(4*1+1)(4*1*3+4+3)}-\frac{1}{(4*1*3+4+3)(4*1*3*2+4*2+3*2+1)}$$

$$=\frac{1}{4}-\frac{1}{4*5}+\frac{1}{5*19}-\frac{1}{19*39}\approx\frac{1}{4}-\frac{1}{20}+\frac{1}{100}-\frac{1}{800}=0.25-0.05+0.01-0.001=0.209$$

A general formula, which allows any values for the numerators, is given below in a transformation to three linear fractions.

2.2j $$\frac{A}{B+\frac{C}{D+\frac{E}{F}}}=\frac{A}{B}-\frac{AC}{B(BD+C)}+\frac{ACE}{(BD+C)(BDF+BE+CF)}$$

2.3 General Square Root Formula

Beginning with the formula in 2.1a and substituting m=1 and n=2 gives

2.3a $r = (y)^{1/2} = (\alpha^2 + \beta)^{1/2} = \alpha + \delta$

Taking the last two parts

2.3b $(\alpha^2 + \beta)^{1/2} = \alpha + \delta$ and transforming gives

2.3c $\alpha^2 + \beta = (\alpha + \delta)^2$

2.3d $\alpha^2 + \beta = \alpha^2 + 2\alpha\delta + \delta^2$. The α^2s cancel creating a formula in terms of δ which is

2.3e $\delta = \frac{\beta}{2\alpha + \delta}$

This is the formula to be used in the iterations and is therefore rewritten as

2.3f $\delta_1 = \frac{\beta}{2\alpha + \delta_0}$

δ_0 is ignored due to its relative size compared to 2α. The second approximation, the first being α, is therefore

2.3g $r_2 = \alpha + \delta_1 = \alpha + \frac{\beta}{2\alpha}$

The second approximation of δ, δ_2 is then

2.3h $\delta_2 = \cfrac{\beta}{2\alpha + \cfrac{\beta}{2\alpha + \delta_0}}$

Ignoring δ_0 gives the next approximation of the root

2.3i $r_3 = \alpha + \delta_2 = \alpha + \cfrac{\beta}{2\alpha + \cfrac{\beta}{2\alpha}}$

A pattern is already apparent and continues through further iterations which results in the following general formula for the square root of Y.

2.3j $$r = (y)^{1/2} = (\alpha^2 + \beta)^{1/2} = \alpha + \cfrac{\beta}{2\alpha + \cfrac{\beta}{2\alpha + \cfrac{\beta}{2\alpha + \cfrac{\beta}{2\alpha + \dots}}}}$$

Using this general formula, to the third fraction, we'll calculate an approximation of the square root of 10. Setting $\alpha = 3$ and $\beta = 1$ gives

2.3k $$(10)^{1/2} = (3^2 + 1)^{1/2} \approx 3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6}}} = 3 + \cfrac{1}{6 + \cfrac{6}{37}} = 3 + \frac{37}{228}$$

Which equals 3.162280701… , correct to 4 decimal places.

Using the linear transformation of continued fractions in 2.2d on the first two fractions gives

$$\cfrac{1}{6 + \cfrac{1}{6}} = \frac{1}{6} - \frac{1}{222}$$

$\frac{1}{222} \approx 0.0045$, 1/200=0.005 and 1/250=0.004

0.166666… - 0.0045 = 0.1621666… which is correct to 5 decimal places of the expected decimal of 6/37 which is 0.162162…

The general formula for square roots calculated in 2.3j will be improved upon in later sections.

2.4 Square Roots of Low Numbers

The accuracy of the roots can be increased considerably if lower values of β relative to α are used. To facilitate this, manipulations of the original number can assist.

In calculating the square root of 2 using 2.3j, setting $\alpha = 1$ and $\beta = 1$ gives the following

2.4a $$(2)^{1/2} = (1^2 + 1)^{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}$$

The approximations are then

1.	1	
2.	1.5	
3.	1.4	correct to 1 decimal place
4.	1.41666…	correct to 2 decimal places
5.	1.413793…	correct to 2 decimal places

These approximations will converge towards the actual answer of 1.414213562… but at a relatively slow rate. Below are three ways that can improve this.

Multiplying the value of y by 4 will result in an answer of twice the root that is required. For the square root of 2, therefore, the initial value of y is taken as 4*2=8. α is set to 3 and β is set to –1. This gives the following

$$r = (8)^{1/2} = 3 - \cfrac{1}{6 - \cfrac{1}{6 - \cfrac{1}{6 - \ldots}}}$$

The approximations of this are

1. 3
2. 2.8333…
3. 2.828571429…
4. 2.828431373…

Dividing these results by 2 gives the following estimates for $2^{1/2}$

1.	1.5	
2.	1.41666…	correct to 2 decimal places
3.	1.414285715…	correct to 4 decimal places
4.	1.414215686…	correct to 5 decimal places

The second method is to multiply the initial value of y by 100 and then divide the result by the square root of 100, which is 10. For 2 this gives y=200, $\alpha = 14$ and $\beta = 4$. Using 2.3j, the general formula, gives

$$r = (200)^{1/2} = (14^2 + 4)^{1/2} = 14 + \cfrac{4}{28 + \cfrac{4}{28 + \cfrac{4}{28 + \ldots}}}$$

The calculated approximations for $(200)^{1/2}$ are therefore

1. 14
2. 14.14285714…
3. 14.142131979…
4. 14.142135642…

Dividing these results by 10 gives the following estimates for $2^{1/2}$

1. 1.4
2. 1.414285714… correct to 4 decimal places
3. 1.4142131979… correct to 6 decimal places
4. 1.4142135642… correct to 8 decimal places

The third method is to take a fractional estimate of α instead of the integer value of 1. If α is set as 3/2, β is then $y - \alpha^2 = 2 - \frac{9}{4} = -\frac{1}{4}$. Substituting these values into 2.3j gives the following

$$r = (2)^{1/2} = \left[\left(\frac{3}{2}\right)^2 - \frac{1}{4}\right]^{1/2} = \frac{3}{2} + \cfrac{\left(\frac{-1}{4}\right)}{2\left(\frac{3}{2}\right) + \cfrac{\left(\frac{-1}{4}\right)}{2\left(\frac{3}{2}\right) + \cfrac{\left(\frac{-1}{4}\right)}{2\left(\frac{3}{2}\right) + \ldots}}}$$

Continued approximations for $2^{1/2}$ are therefore

1. 1.5
2. 1.41666… correct to 2 decimal places
3. 1.414285714… correct to 4 decimal places
4. 1.414215686… correct to 5 decimal places

These results are the same as those encountered in the first example, which initially took the square root of 8.

2.5 Cube Roots by Example $\sqrt[3]{65}$

The actual answer is 4.020725759…

The process is similar to the approach in 2.1. We begin by taking the initial formula in the introduction

2.5a $r = (y)^{m/n} = (\alpha^n + \beta)^{m/n} = \alpha^m + \delta$

The cube root of 65 will lie between 4 and 5. 4^3 being 64 and 5^3 being 125. Assigning the appropriate values we have $y = 65, m = 1, n = 3, \alpha = 4, \beta = 1$. δ will lie in the range of 0 to 1. Substituting these values into 2.5a gives

2.5b $r = (65)^{1/3} = (4^3 + 1)^{1/3} = 4 + \delta$

The aim here is to create a formula that gives a value for δ. Working with the last two parts of 2.5b gives

2.5c $(4^3 + 1)^{1/3} = 4 + \delta$

2.5d $4^3 + 1 = (4 + \delta)^3$

2.5e $64 + 1 = 64 + 48\delta + 12\delta^2 + \delta^3$, cancelling the 64s and rearranging gives

2.5f $1 = \delta(48 + 12\delta + \delta^2)$, and then

2.5g $\delta = \dfrac{1}{48 + 12\delta + \delta^2}$

This formula will be used iteratively to calculate closer approximations to the true value of δ and is therefore rewritten as

2.5h $\delta_1 = \dfrac{1}{48 + 12\delta_0 + \delta_0{}^2}$

δ_0 is relatively small and is therefore ignored. The estimate of δ_1 is therefore

2.5i $\delta_1 = \dfrac{1}{48}$

Our second root estimate, the first being our initial estimate for α of 4, will be

$r_2 = \alpha + \delta_1 = 4 + \dfrac{1}{48} = 4.0208333...$ which is correct to 3 decimal places.

The next approximation of δ is achieved by substituting the calculated value of δ_1 into 2.5h in place of δ_0. Which gives

2.5j $$\delta_2 = \frac{1}{48 + 12\left(\frac{1}{48 + 12\delta_0 + \delta_0^{\ 2}}\right) + \left(\frac{1}{48 + 12\delta_0 + \delta_0^{\ 2}}\right)^2}$$

δ_0 is again ignored, because of its relative size, giving

2.5k $$\delta_2 = \frac{1}{48 + 12\left(\frac{1}{48}\right) + \left(\frac{1}{48}\right)^2} = \frac{1}{48 + \frac{1}{4} + \frac{1}{2304}} = 0.020725202...$$

The third approximation of the root is therefore

$r_3 = \alpha + \delta_2 = 4 + 0.020725202 = 4.020725202$, correct to 5 decimal places. The last fraction in the denominator, 1/2304, could be dropped as its impact on the overall result is minimal, giving 4.020725388…

The next approximation of δ, δ_3 is achieved in the same way by substituting the value of δ_2 in 2.5k into 2.5h replacing δ_0. After removing the values of δ_0 this gives

2.5l $$\delta_3 = \frac{1}{48 + 12\left(\frac{1}{48 + 12\left(\frac{1}{48}\right) + \left(\frac{1}{48}\right)^2}\right) + \left(\frac{1}{48 + 12\left(\frac{1}{48}\right) + \left(\frac{1}{48}\right)^2}\right)^2}$$

$$\delta_3 = 0.020725761...$$

The fourth approximation of the root is then

$r_4 = \alpha + \delta_3 = 4 + 0.020725761 = 4.020725761$, correct to 7 decimal places.

This can be continued to any degree, and will see a gain of approximately 2 decimal places for each additional iteration. The gain of 2 decimal places is only applicable to this specific calculation. Other numbers will converge at different rates. This is still only the beginning. In later sections a method will be described that will increase this rate of convergence.

The next section derives a general formula for cube roots that will simplify the substitutions of δ_0 used in this example.

2.6 General Cube Root Formula

Beginning with the formula in 2.5a and substituting m=1 and n=3 gives

2.6a $r = (y)^{1/3} = (\alpha^3 + \beta)^{1/3} = \alpha + \delta$

Taking the last two parts

2.6b $(\alpha^3 + \beta)^{1/3} = \alpha + \delta$ and transforming gives

2.6c $\alpha^3 + \beta = (\alpha + \delta)^3$

2.6d $\alpha^3 + \beta = \alpha^3 + 3\alpha^2\delta + 3\alpha\delta^2 + \delta^3$. The α^3s cancel creating a formula in terms of δ which is

2.6e $$\delta = \frac{\beta}{3\alpha^2 + 3\alpha\delta + \delta^2}$$

This is the formula to be used in the iterations and is therefore rewritten as

2.6f $$\delta_1 = \frac{\beta}{3\alpha^2 + 3\alpha\delta_0 + \delta_0{}^2}$$

For the second approximation of the root, r_2, δ_0 again being ignored, is

2.6g $$r_2 = \alpha + \delta_1 = \alpha + \frac{\beta}{3\alpha^2}$$

The second approximation of δ, δ_2 is then

2.6h $$\delta_2 = \frac{\beta}{3\alpha^2 + 3\alpha\left(\frac{\beta}{3\alpha^2 + 3\alpha\delta_0 + \delta_0{}^2}\right) + \left(\frac{\beta}{3\alpha^2 + 3\alpha\delta_0 + \delta_0}\right)^2}$$

Ignoring δ_0 simplifies the equation to

2.6i $$\delta_2 = \frac{\beta}{3\alpha^2 + 3\alpha\left(\frac{\beta}{3\alpha^2}\right) + \left(\frac{\beta}{3\alpha^2}\right)^2} = \frac{\beta}{3\alpha^2 + \frac{\beta}{\alpha} + \frac{\beta^2}{9\alpha^4}}$$

As with all the general methods, only the first new fraction will be taken, giving our new estimate as

2.6j $$\delta_2 = \frac{\beta}{3\alpha^2 + \frac{\beta}{\alpha}}$$

The third approximation is then

2.6k $r_3 = \alpha + \delta_2 = \alpha + \cfrac{\beta}{3\alpha^2 + \cfrac{\beta}{\alpha}}$

Which can be rewritten as

2.6l $r_3 = \alpha + \dfrac{\alpha\beta}{3\alpha^3 + \beta}$

From 2.6a, knowing that $y = \alpha^3 + \beta$, the denominator can be rewritten as $3y - 2\beta$, resulting in

2.6m $r_3 = \alpha + \dfrac{\alpha\beta}{3y - 2\beta}$

Using this on $(65)^{1/3}$, the example in 2.5, gives $y = 65, \alpha = 4, \beta = 1$

$r = (4) + \dfrac{(4)(1)}{3(65) - 2(1)} = 4 + \dfrac{4}{193} = 4.020725388...$ correct to 6 decimal places.

The equation in 2.6m is an example of a folded fraction, where 2 continuous fractions are folded together to create 1. This will become useful in the latter stages as we derive a folded general formula.

The approximations can be recalculated by further iterations to produce a result even closer to the actual root. I will go through this process for the next iteration of cube roots not only to derive the next fraction but to also highlight the advantage of having a general solution that avoids all these interim steps.

The next approximation of δ is calculated by substituting the value in 2.6j into 2.6f in place of δ_0

2.6n $\delta_3 = \cfrac{\beta}{3\alpha^2 + \cfrac{3\alpha\beta}{3\alpha^2 + \cfrac{\beta}{\alpha} + \cfrac{\beta^2}{9\alpha^4}} + \cfrac{\beta^2}{9\alpha^4 + 6\alpha\beta + \cfrac{5\beta^2}{3\alpha^2} + \cfrac{2\beta^2}{9\alpha^5} + \cfrac{\beta^4}{81\alpha^8}}}$

$$= \cfrac{\beta}{3\alpha^2 + \cfrac{3\alpha\beta\left(3\alpha^2 + \cfrac{\beta}{\alpha} + \cfrac{\beta^2}{9\alpha^4}\right) + \beta^2}{9\alpha^4 + 6\alpha\beta + \cfrac{5\beta^2}{3\alpha^2} + \cfrac{2\beta^3}{9\alpha^5} + \cfrac{\beta^4}{81\alpha^8}}}$$

$$= \cfrac{\beta}{3\alpha^2 + \cfrac{9\alpha^3\beta + 4\beta^2 + \cfrac{\beta^3}{3\alpha^3}}{9\alpha^4 + 6\alpha\beta + \cfrac{5\beta^2}{3\alpha^2} + \cfrac{2\beta^3}{9\alpha^5} + \cfrac{\beta^4}{81\alpha^8}}}$$

$$= \frac{\beta}{3\alpha^2 + \cfrac{\beta\left(9\alpha^3 + 4\beta + \cfrac{\beta^2}{3\alpha^3}\right)}{9\alpha^4 + 6\alpha\beta + \cfrac{5\beta^2}{3\alpha^2} + \cfrac{2\beta^3}{9\alpha^5} + \cfrac{\beta^4}{81\alpha^8}}}$$

$$= \frac{\beta}{3\alpha^2 + \cfrac{\beta}{\cfrac{9\alpha^4 + 6\alpha\beta + \cfrac{5\beta^2}{3\alpha^2} + \cfrac{2\beta^3}{9\alpha^5} + \cfrac{\beta^4}{81\alpha^8}}{9\alpha^3 + 4\beta + \cfrac{\beta^2}{3\alpha^3}}}}$$

$$= \frac{\beta}{3\alpha^2 + \cfrac{\beta}{\alpha + \cfrac{2\beta}{9\alpha^2} + \cfrac{4\beta^2}{81\alpha^5} - \cfrac{4\beta^3}{729\alpha^8}....}}$$

For simplicity, as always, only the first new fraction, $\frac{2\beta}{9\alpha^2}$, is used, giving

2.6o $$\delta_3 = \frac{\beta}{3\alpha^2 + \cfrac{\beta}{\alpha + \cfrac{2\beta}{9\alpha^2}}}$$

The next approximation of the root, r_4, is then

2.6p $$r_4 = \alpha + \delta_3 = \alpha + \frac{\beta}{3\alpha^2 + \cfrac{\beta}{\alpha + \cfrac{2\beta}{9\alpha^2}}}$$

Using our previous example of $y = 65, \alpha = 4, \beta = 1$ gives

$$\sqrt[3]{65} \approx 4 + \frac{1}{3(4)^2 + \cfrac{1}{4 + \cfrac{2(1)}{9(4)^2}}} = 4 + \frac{1}{48 + \cfrac{1}{4 + \cfrac{2}{144}}} = 4 + \frac{1}{48 + \cfrac{144}{578}} = 4 + \frac{578}{27888} = 4.0207257601...$$

An answer that is correct to 7 decimal places.

Substitutions of δ_0 can be continued giving the following general solution for $(y)^{1/3}$

2.6q $$r = (y)^{1/3} = (\alpha^3 + \beta)^{1/3}$$

$$= \alpha + \cfrac{\beta}{3\alpha^2 + \cfrac{\beta}{\alpha + \cfrac{2\beta}{9\alpha^2 + \cfrac{5\beta}{2\alpha + \cfrac{7\beta}{15\alpha^2 + \cfrac{4\beta}{\alpha + \ldots}}}}}}$$

After deriving the complete general formula in a later section, a pattern will emerge in the sequence of fractions. Until then here is the extended result of $\sqrt[3]{65}$ to the fourth fraction

$$r = 4 + \cfrac{1}{48 + \cfrac{1}{4 + \cfrac{2}{144 + \cfrac{5}{8}}}}$$

This gives the following approximations compared to the actual value of 4.0207257585884…

1.	4	
2.	4.0208333…	correct to 3 decimal places
3.	4.020725388…	correct to 6 decimal places
4.	4.020725760183…	correct to 7 decimal places
5.	4.020725758583…	correct to 11 decimal places

This method will be improved upon in the later section containing the general folded method.

2.7 General Formula for Roots of the Form $(y)^{1/n}$

As always, beginning with our initial formula and then substituting m=1 gives

2.7a $r = (y)^{1/n} = (\alpha^n + \beta)^{1/n} = \alpha + \delta$

Taking the last two parts

2.7b $(\alpha^n + \beta)^{1/n} = \alpha + \delta$ and transforming gives

2.7c $\alpha^n + \beta = (\alpha + \delta)^n$

2.7d $\alpha^n + \beta = \alpha^n + \frac{n\alpha^{(n-1)}\delta}{1!} + \frac{n(n-1)\alpha^{(n-2)}\delta^2}{2!} + \frac{n(n-1)(n-2)\alpha^{(n-2)}\delta^3}{3!} + \ldots$. The α^ns cancel.

Rearranging in terms of δ gives

2.7e $$\delta = \frac{\beta}{n\alpha^{(n-1)} + \frac{n(n-1)\alpha^{(n-2)}\delta}{2!} + \frac{n(n-1)(n-2)\delta^2}{3!} + \ldots}$$

Taking the first term in the denominator gives our first estimate

2.7f $$\delta_1 = \frac{\beta}{n\alpha^{(n-1)}}$$

This leads to the second root estimate

2.7g $$r_2 = \alpha + \delta_1 = \alpha + \frac{\beta}{n\alpha^{(n-1)}}$$

Which agrees nicely with our two previous examples of square root, $\alpha + \frac{\beta}{2\alpha}$, and cube root, $\alpha + \frac{\beta}{3\alpha^2}$.

Substituting the value of δ_1, found in 2.7f, into 2.7e gives the next estimate for δ.

2.7h $$\delta = \frac{\beta}{\frac{n\alpha^{(n-1)}}{1!} + \frac{n(n-1)\alpha^{(n-2)}\left(\frac{\beta}{n\alpha^{(n-1)}}\right)}{2!} + \frac{n(n-1)(n-2)\left(\frac{\beta}{n\alpha^{(n-1)}}\right)^2}{3!} + \ldots}$$

As previously, taking only the second additional fraction and reducing gives

2.7i $$\delta_2 = \frac{\beta}{n\alpha^{(n-1)} + \frac{\beta(n-1)}{2\alpha}}$$

The third approximation to the root is then

2.7j $$r_3 = \alpha + \delta_2 = \alpha + \cfrac{\beta}{n\alpha^{(n-1)} + \cfrac{\beta(n-1)}{2\alpha}}$$

Agreeing again with our previous examples.

Square root to second fraction is $\alpha + \cfrac{\beta}{2\alpha + \cfrac{\beta}{2\alpha}}$

Cube root to second fraction is $\alpha + \cfrac{\beta}{3\alpha^2 + \cfrac{\beta}{\alpha}}$

This process of substitution can be continued to develop the following general formula.

2.7k $$r = (y)^{1/n} = (\alpha^n + \beta)^{1/n}$$

$$= \alpha + \cfrac{\beta}{n\alpha^{(n-1)} + \cfrac{\beta(n-1)}{2\alpha + \cfrac{\beta(n+1)}{3n\alpha^{(n-1)} + \cfrac{\beta(2n-1)}{2\alpha + \cfrac{\beta(2n+1)}{5n\alpha^{(n-1)} + \cfrac{\beta(3n-1)}{2\alpha + \cfrac{\beta(3n+1)}{7n\alpha^{(n-1)} + \ldots}}}}}}}$$

There is an underlying pattern in both the numerators and denominators which is evident from the second fraction onwards. Numerators are of the form xn+1 or xn-1 and increase steadily by the addition of another multiple of n. Denominators are alternately 2α and $xn\alpha^{(n-1)}$, where x, in this case takes the value of increasing odd numbers.

This can now be used to create a formula for any root of the form $(y)^{1/n}$. A general 5th root would develop from substituting the value of n with 5, and arrives at the following

2.7l $$r = (y)^{1/5} = (\alpha^5 + \beta)^{1/5}$$

$$= \alpha + \cfrac{\beta}{5\alpha^4 + \cfrac{4\beta}{2\alpha + \cfrac{6\beta}{15\alpha^4 + \cfrac{9\beta}{2\alpha + \cfrac{11\beta}{25\alpha^4 + \cfrac{14\beta}{2\alpha + \ldots}}}}}}$$

The 5th root of 33 would then have the following equivalent continued fraction. Setting $y = 33, \alpha = 2, \beta = 1$

$$r = (33)^{1/5} = (2^5 + 1)^{1/5}$$

$$= 2 + \cfrac{1}{80 + \cfrac{4}{4 + \cfrac{6}{240 + \cfrac{9}{4 + \cfrac{11}{400 + \cfrac{14}{4 + \ldots}}}}}}$$

This gives the following approximations compared to the actual value of 2.0123466170853…

1. 2
2. 2.0125 correct to 3 decimal places
3. 2.012345679… correct to 5 decimal places
4. 2.012346625… correct to 7 decimal places
5. 2.012346617026… correct to 10 decimal places

This method will be improved upon in the later sections.

3. ROOTS OF THE FORM $(y)^{m/n}$

This section deals with fractional roots. For example $(64)^{2/3}$ means the cube root of 64 squared.

$64 = 4^3 \rightarrow (64)^{2/3} = (4^3)^{2/3} = 4^2 = 16$.

3.1 Two-Third Roots by Example $(65)^{2/3}$

The actual answer is 16.16623563…

We begin by taking our standard basic formula and substituting the values of m=2 and n=3. My thinking is that, if I look at the cube root first, then I am saying there is an integer root a and the left over decimal part x of the equation. so the initial number is therefore a^3 + b. If I take the cube root of this and then square it, the answer will have a part made up of a^2 the rest of the number I have labelled d.

3.1a $r = (y)^{2/3} = (\alpha^3 + \beta)^{2/3} = \alpha^2 + \delta$

First we need to choose a value for α. From the equation above we can see we are looking for a number, such that, when it is cubed it is close to the value of y, which in this case is 65. $4^3 = 65$ and $5^3 = 125$. So a convenient value for α is 4. The root of 65 will then lie between $4^2 = 16$ and $5^2 = 25$. Assigning the appropriate values we have $y = 65, m = 2, n = 3, \alpha = 4, \beta = 1$. Substituting these values into 3.1a gives

3.1b $r = (65)^{2/3} = (4^3 + 1)^{2/3} = 4^2 + \delta$

The aim here is to create a formula that gives a value for δ. Working with the last two parts of 3.1b gives

3.1c $(4^3 + 1)^{2/3} = 4^2 + \delta$

3.1d $(4^3 + 1)^2 = (4^2 + \delta)^3$

3.1e $4225 = 4096 + 768\delta + 48\delta^2 + \delta^3$

3.1f $129 = \delta(768 + 48\delta + \delta^2)$

3.1g $\delta = \frac{129}{768 + 48\delta + \delta^2}$

This formula will be used iteratively to calculate closer approximations to the true value of δ and is therefore rewritten as

3.1h $\delta_1 = \frac{129}{768 + 48\delta_0 + \delta_0^{\,2}}$

δ_0 is relatively small and is therefore ignored. The estimate of δ_1 is therefore

3.1i $\delta_1 = \frac{129}{768}$

Our second root estimate will be

$r_2 = \alpha^2 + \delta_1 = 16 + \frac{129}{768} = 16.16796875...$ which is correct to 2 decimal places.

The next approximation of δ is achieved by substituting the calculated value of δ_1 into 3.1h in place of δ_0. Which gives

3.1j $$\delta_2 = \frac{129}{768 + 48\left(\frac{129}{768 + 48\delta_0 + \delta_0^2}\right) + \left(\frac{129}{768 + 48\delta_0 + \delta_0^2}\right)^2}$$

δ_0 is again ignored, because of its relative size, giving

3.1k $$\delta_2 = \frac{129}{768 + 48\left(\frac{129}{768}\right) + \left(\frac{129}{768}\right)^2} = 0.166217682...$$

The third approximation of the root is therefore

$r_3 = \alpha + \delta_2 = 16 + 1.0166217682 = 16.166217682$, correct to 4 decimal places. The last fraction in the denominator, (129/768)^2, could be dropped here as it has little impact on the result, giving 0.166223725…

Further closer approximations of δ can be derived by replacing δ_0 in 3.1h with the current value of δ.

3.2 General Formula for Roots of the Form $(y)^{2/3}$

At this stage we take the original formula the values of m=2 and n=3.

3.2a $r = (y)^{2/3} = (\alpha^3 + \beta)^{2/3} = \alpha^2 + \delta$

Taking the last two parts

3.2b $(\alpha^3 + \beta)^{2/3} = \alpha^2 + \delta$ and transforming, gives

3.2c $(\alpha^3 + \beta)^2 = (\alpha^2 + \delta)^3$

3.2d $\alpha^6 + 2\alpha^3\beta + \beta^2 = \alpha^6 + 3\alpha^4\delta + 3\alpha^2\delta^2 + \delta^3$

The first terms cancel, transforming to give a value for δ of

3.2e $\delta = \dfrac{2\alpha^3\beta + \beta^2}{3\alpha^4 + 3\alpha^2\delta + \delta^2}$

This formula will be used iteratively to calculate closer approximations of δ and is therefore written as

3.2f $\delta_1 = \dfrac{2\alpha^3\beta + \beta^2}{3\alpha^4 + 3\alpha^2\delta_0 + \delta_0{}^2}$

The first estimate of δ_1, ignoring multiples of δ_0, is

3.2g $\delta_1 = \dfrac{2\alpha^3\beta + \beta^2}{3\alpha^4} = \dfrac{2\beta + \dfrac{\beta^2}{\alpha^3}}{3\alpha}$

Due to its relative size the last term in the numerator is removed giving

3.2h $\delta_1 = \dfrac{2\beta}{3\alpha}$

This leads to the second root estimate, r_2, of

3.2i $r_2 = \alpha^2 + \dfrac{2\beta}{3\alpha}$

For the next estimate of δ, the value of δ_1 is substituted into 3.2f in place of δ_0, giving

3.2j $\delta_2 = \dfrac{2\alpha^3\beta + \beta^2}{3\alpha^4 + 3\alpha^2\left(\dfrac{2\beta}{3\alpha}\right) + \left(\dfrac{2\beta}{3\alpha}\right)^2}$

$$= \frac{2\alpha^3\beta + \beta^2}{3\alpha^4 + 2\alpha\beta + \frac{4\beta^2}{9\alpha^2}} = \frac{18\alpha^5\beta + 9\alpha^2\beta^2}{27\alpha^6 + 18\alpha^3\beta + 4\beta^2}$$ we now work back to the formula in 3.2h

$$= \frac{2\beta\left(9\alpha^5 + \frac{9\alpha^2\beta}{2}\right)}{27\alpha^6 + 18\alpha^3\beta + 4\beta^2} = \frac{2\beta}{\frac{27\alpha^6 + 18\alpha^3\beta + 4\beta^2}{9\alpha^5 + \frac{9\alpha^2\beta}{2}}} = \frac{2\beta}{\frac{54\alpha^6 + 36\alpha^3\beta + 8\beta^2}{18\alpha^5 + 9\alpha^2\beta}} = \frac{2\beta}{3\alpha + \frac{9\alpha^3\beta + 8\beta^2}{18\alpha^5 + 9\alpha^2\beta}}$$

$$= \frac{2\beta}{3\alpha + \frac{\beta}{2\alpha^2}}$$ if we take only the first terms in the new fraction.

This leads to the third root estimate, r_3, of

3.2k $$r_3 = \alpha^2 + \frac{2\beta}{3\alpha + \frac{\beta}{2\alpha^2}}$$

This can be folded to give

3.2l $$r_3 = \alpha^2 + \frac{4\alpha^2\beta}{6\alpha^3 + \beta}$$

Given that $y = \alpha^3 + \beta$ this can be rewritten as

3.2m $$r_3 = \alpha^2 + \frac{4\alpha^2\beta}{6y - 5\beta}$$

An estimate of $(65)^{2/3}$ would then be

$$r = (65)^{2/3} = (4^3 + 1)^{2/3} = 4^2 + \frac{4(4)^2(1)}{6(65) - 5(1)} = 16 + \frac{64}{385} = 16.166233766\ldots$$ correct to 5 decimal places.

Further expansion of this can be derived from the general formula in the next section.

3.3 General Formula for Roots of the Form $(y)^{m/n}$

At this stage we take the original formula in the introduction in its purest form

3.3a $\quad r = (y)^{m/n} = (\alpha^n + \beta)^{m/n} = \alpha^m + \delta$

Taking the last two parts

3.3b $\quad (\alpha^n + \beta)^{m/n} = \alpha^m + \delta$ and transforming gives

3.3c $\quad (\alpha^n + \beta)^m = (\alpha^m + \delta)^n$

3.3d $$\alpha^{mn} + \frac{m\alpha^{n(m-1)}\beta}{1!} + \frac{m(m-1)\alpha^{n(m-2)}\beta^2}{2!} + \ldots = \alpha^{mn} + \frac{n\alpha^{m(n-1)}\delta}{1!} + \frac{n(n-1)\alpha^{m(n-2)}\delta^2}{2!} + \ldots.$$

The α^{mn} s cancel.

Transforming to give a value for δ_1 results in

3.3e $$\delta_1 = \frac{\dfrac{m\alpha^{n(m-1)}\beta}{1!} + \dfrac{m(m-1)\alpha^{n(m-2)}\beta^2}{2!} + \dfrac{m(m-1)(m-2)\alpha^{n(m-3)}\beta^3}{3!} + \ldots}{\dfrac{n\alpha^{m(n-1)}}{1!} + \dfrac{n(n-1)\alpha^{m(n-2)}\delta_0}{2!} + \dfrac{n(n-1)(n-2)\alpha^{m(n-3)}\delta_0{}^2}{3!} + \ldots}$$

Taking the first terms in both the numerator and denominator gives our first estimate

3.3f $$\delta_1 = \frac{m\alpha^{n(m-1)}\beta}{n\alpha^{m(n-1)}} = \frac{m\beta}{n\alpha^{(n-m)}}$$

The second root estimate, r_2, is then given by

3.3g $$r_2 = \alpha^m + \delta_1 = \alpha^m + \frac{m\beta}{n\alpha^{(n-m)}}$$

Through further development this results in the general formula

3.3h $\quad r = (y)^{m/n} = (\alpha^n + \beta)^{m/n} = \alpha^m + \delta$

$$= \alpha^m + \cfrac{\beta m}{n\alpha^{(n-m)} + \cfrac{\beta(n-m)}{2\alpha^m + \cfrac{\beta(n+m)}{3n\alpha^{(n-m)} + \cfrac{\beta(2n-m)}{2\alpha^m + \cfrac{\beta(2n+m)}{5n\alpha^{(n-m)} + \cfrac{\beta(3n-m)}{2\alpha^m + \cfrac{\beta(3n+m)}{7n\alpha^{(n-m)} + \ldots}}}}}}}$$

There is a pattern that becomes apparent if we look at alternate fractions from the second onwards. Numerators increase in multiples of n at every other fraction. Denominators alternate with $2\alpha^m$ and consecutive odd multiples of $n\alpha^{(n-m)}$.

This general method will be improved upon in the next section.

4. FOLDED FRACTIONS

This section looks at increasing the convergence rate by combining consecutive pairs of fractions in the general formula given in 3.3h.

4.1 Method of Folding Fractions

Taking the simple case of

4.1a $$\frac{A}{B+\frac{C}{D}}$$ this can be folded, multiplied upwards, to produce

4.1b $$\frac{AD}{BD+C}$$

The next stage is to add two more fractions and fold them

4.1c

$$\frac{A}{B+\frac{C}{D+\frac{E}{F+\frac{G}{H}}}}=\frac{A}{B+\frac{C}{D+\frac{EH}{FH+G}}}=\frac{A}{B+\frac{CFH+CG}{DFH+DG+EH}}=\frac{ADFH+ADG+AEH}{BDFH+BDG+BEH+CFH+CG}$$

From 4.1b the first numerator is to be AD. Factoring the numerator, therefore, by AD gives

4.1d $$\frac{AD\left(FH+G+\frac{EH}{D}\right)}{BDFH+BDG+BEH+CFH+CG}$$

4.1e $$\frac{AD}{\frac{BDFH+BDG+BEH+CFH+CG}{FH+G+\frac{EH}{D}}}$$

From 4.1b the first part of the first denominator is to be BD+C. Multiplying this by the new denominator, $FH+G+\frac{EH}{D}$, is

$$BDFH+BDG+BEH+CFH+CG+\frac{CEH}{D}$$

Subtracting this from the first denominator, or middle line, in 4.1e gives

$$-\frac{CEH}{D}$$

The current formula, for four fractions folded to two, is now

4.1f
$$\frac{AD}{BD+C-\frac{\frac{CEH}{D}}{FH+G+\frac{EH}{D}}}=\frac{AD}{BD+C-\frac{CEH}{DFH+DG+EH}}$$

Expanding this idea to six fractions and then folding to three results in

4.1g
$$\frac{A}{B+\frac{C}{D+\frac{E}{F+\frac{G}{H+\frac{I}{J+\frac{K}{L}}}}}}=\frac{AD}{BD+C-\frac{CEH}{DFH+DG+EH-\frac{DGIL}{HJL+HK+IL}}}$$

This methodology can be applied to further extensions of continued fractions.

The next section applies this method to the general formula given in 3.3h resulting in an even more powerful tool for deriving roots.

4.2 General Folded Fractional Roots

If we take the first two fractions of the continued fraction in 3.3h we can fold them as follows

4.2a $$\frac{\beta m}{n\alpha^{(n-m)} + \dfrac{\beta(n-m)}{2\alpha^m}} = \frac{2\alpha^m \beta m}{2n\alpha^n + \beta(n-m)}$$

Knowing, from 3.3h, that $y = \alpha^n + \beta$, this can be rewritten as

4.2b $$\frac{2\alpha^m \beta m}{2ny - \beta(m+n)}$$

The first folded fractional root is therefore

4.2c $$\alpha^m + \frac{2\alpha^m \beta m}{2ny - \beta(m+n)}$$

By using the folding formula in 4.1g, the first six fractions of the general formula in 3.3h can be reworked to give the first three folded fractions. The resultant root can therefore be written as

4.2d $$r = \alpha^m + \cfrac{2\alpha^m \beta m}{2ny - \beta(m+n) - \cfrac{(n^2 - m^2)\beta^2}{6ny - 3n\beta - \cfrac{(4n^2 - m^2)\beta^2}{10ny - 5n\beta}}}$$

This can be extended by developing the idea in the previous section on folded fractions, applying it to a continuation of the general formula in 3.3h. Regrouping the results as follows brings to light a convenient pattern and the general folded formula that we have been working towards.

4.2e $$r = (y)^{m/n} = (\alpha^n + \beta)^{m/n} = \alpha^m + \delta$$

$$= \alpha^m + \cfrac{2\alpha^m \beta m}{2ny - \beta(m+n) - \cfrac{(n^2 - m^2)\beta^2}{3n(2y-\beta) - \cfrac{(4n^2 - m^2)\beta^2}{5n(2y-\beta) - \cfrac{(9n^2 - m^2)\beta^2}{7n(2y-\beta) - \cfrac{(16n^2 - m^2)\beta^2}{9n(2y-\beta) - \cfrac{(25n^2 - m^2)\beta^2}{11n(2y-\beta) - \dots}}}}}}$$

From the second fraction onwards, the multiples of n^2 in the numerators take the next square number. The multiples of n in the denominators increase to the next odd number.
Because of the way that it is made, this folded formula will tend towards the actual root twice as quickly as the unfolded version given in 3.3h.

In section 6 the rate of convergence to the actual root is improved upon by giving suggestions as to the best values of α and β to use for a particular combination of y, m and n.

4.3 Approach to Actual Roots

This section is only to highlight an interesting feature of the folded fraction solution in 4.2e, in that subsequent approximations approach the true value from only one direction.

For example $\sqrt{10}$, setting $y = 10, \alpha = 3, \beta = 1, m = 1, n = 2$ into 4.2e gives

$$r = 3 + \cfrac{6}{37 - \cfrac{3}{114 - \cfrac{15}{190 - \cfrac{35}{266...}}}}$$

This has the resulting approximations compared to the actual answer of 3.16227766016837933…

1. 3
2. 3.162162162… correct to 3 decimal places
3. 3.16227758007117… correct to 6 decimal places
4. 3.16227766011283… correct to 10 decimal places
5. 3.16227766016834… correct to 13 decimal places

The estimates are constantly increasing, as will be the case if β is positive. With a negative value of β the estimates constantly decrease towards the root.

5. SPECIAL CASE OF SQUARE ROOTS

5.1 Reduction of General Folded Formula for Square Roots

Square roots offer further reductions which assist in the extraction of roots.

The general folded formula in 4.2e substituted with m=1 an n=2 gives

5.1a $r = (y)^{1/2} = (\alpha^2 + \beta)^{1/2} = \alpha + \delta$

$$= \alpha + \cfrac{2\alpha\beta}{4y - 3\beta - \cfrac{3\beta^2}{12y - 6\beta - \cfrac{15\beta^2}{20y - 10\beta - \cfrac{35\beta^2}{28y - 14\beta - \cfrac{63\beta^2}{36y - 18\beta - \cfrac{99\beta^2}{44y - 22\beta - \ldots}}}}}}$$

This can be reduced giving an obvious repeat of the second fraction

5.1b $$r = \alpha + \cfrac{2\alpha\beta}{4y - 3\beta - \cfrac{\beta^2}{4y - 2\beta - \cfrac{\beta^2}{4y - 2\beta - \cfrac{\beta^2}{4y - 2\beta - \cfrac{\beta^2}{4y - 2\beta - \cfrac{\beta^2}{4y - 2\beta - \ldots}}}}}}$$

$\sqrt{10}$ would then be given by

$$r = 3 + \cfrac{6}{37 - \cfrac{1}{38 - \cfrac{1}{38 - \cfrac{1}{38 - \cfrac{1}{38 - \cfrac{1}{38 - \ldots}}}}}}$$

This gives the same results as those found in 4.3 but with simpler fractions. Approximations are as follows

1. 3
2. 3.162162162… correct to 3 decimal places
3. 3.16227758007117… correct to 6 decimal places
4. 3.16227766011283… correct to 10 decimal places
5. 3.16227766016834… correct to 13 decimal places

5.2 Further Reduction of Continued Fractions

$\sqrt{14}$ can be calculated as follows

$y = 14, \alpha = 4, \beta = -2$

Substituting these values into 5.1b gives

$$r = (14)^{1/2} = \left(4^2 - 2\right)^{1/2} = 4 - \cfrac{16}{62 - \cfrac{4}{60 - \cfrac{4}{60 - \cfrac{4}{60 - \ldots}}}}$$

$\frac{16}{62}$ can be reduced to $\frac{8}{31}$ quite easily by dividing through by 2. Reducing 62 however means that the 4 on the same line also needs to be reduced by a factor of 2 giving

$$r = 4 - \cfrac{8}{31 - \cfrac{2}{60 - \cfrac{4}{60 - \cfrac{4}{60 - \ldots}}}}$$

$\frac{2}{60}$ can be reduced to $\frac{1}{30}$. In the same way as before, the 4 on line three needs to be reduced by a factor of 2 giving

$$r = 4 - \cfrac{8}{31 - \cfrac{1}{30 - \cfrac{2}{60 - \cfrac{4}{60 - \ldots}}}}$$

This idea can be continued dividing the continued fraction as follows

$$\cfrac{\div 2}{\div 2 \cfrac{\div 4}{\div 2 \cfrac{\div 4}{\div 2 \cfrac{\div 4}{\div 2 \ldots}}}}$$

It can be seen that the subsequent numerators are divided by the square of the divisor of the denominators.

For our example of 14 we can therefore transform the result into the following

$$r = (14)^{1/2} = 4 - \cfrac{8}{31 - \cfrac{1}{30 - \cfrac{1}{30 - \cfrac{1}{30 - \dots}}}}$$

Here's another example using $\sqrt{12}$

$$r = (12)^{1/2} = (3^2 + 3)^{1/2} = 3 + \cfrac{18}{39 - \cfrac{9}{42 - \cfrac{9}{42 - \cfrac{9}{42 - \dots}}}}$$

This can be reduced by dividing the denominators by 3 and dividing the numerators by 9, 3^2 . The first denominator is only divided by 3. This gives

$$r = (12)^{1/2} = (3^2 + 3)^{1/2} = 3 + \cfrac{6}{13 - \cfrac{1}{14 - \cfrac{1}{14 - \cfrac{1}{14 - \dots}}}}$$

This gives the following approximations compared to the actual answer of 3.4641016151377…

1.	3	
2.	3.4615…	correct to 2 decimal places
3.	3.464088…	correct to 3 decimal places
4.	3.46410154…	correct to 6 decimal places
5.	3.4641016147…	correct to 8 decimal places

5.3 Convergence of the Square Root Formula

It can be shown that for square roots there is a gain of approximately

5.3a $2\log\left(\frac{4y}{|\beta|}-2\right)$ decimal places with each additional folded fraction

From this it can be seen that smaller values of β relative to y yield a more rapid convergence towards the root and methods such as those in 2.4 should be used to optimize these values.

Here are some examples

$\sqrt{10}$ would then be given by $y=10, \alpha=3, \beta=1$

$2\log\left(\frac{4y}{|\beta|}-2\right)=2\log\left(\frac{40}{1}-2\right)=3.159...$ so a net gain of approximately 3 decimal places will be seen for each additional folded fraction used. Working this through gives

$$r=3+\cfrac{6}{37-\cfrac{1}{38-\cfrac{1}{38-\cfrac{1}{38-\cfrac{1}{38-\cfrac{1}{38-...}}}}}}$$

Approximations are as follows

1. 3
2. 3.162162162… correct to 3 decimal places
3. 3.16227758007117… correct to 6 decimal places
4. 3.16227766011283… correct to 10 decimal places
5. 3.16227766016834… correct to 13 decimal places

$\sqrt{12}$ would then be given by $y=12, \alpha=3, \beta=3$

$2\log\left(\frac{4y}{|\beta|}-2\right)=2\log\left(\frac{48}{3}-2\right)=2.292...$ so a net gain of approximately 2 decimal places will be seen for each additional folded fraction used. Working this through gives

$$r = 3 + \cfrac{6}{13 - \cfrac{1}{14 - \cfrac{1}{14 - \cfrac{1}{14 - \cfrac{1}{14 - \cfrac{1}{14 - \ldots}}}}}}$$

Approximations are as follows

6. 3
7. 3.461538461… correct to 2 decimal places
8. 3.464088397… correct to 3 decimal places
9. 3.464101547… correct to 6 decimal places
10. 3.464101614786… correct to 8 decimal places

$\sqrt{14}$ would then be given by $y = 14, \alpha = 4, \beta = -2$

$2\log\left(\frac{4y}{|\beta|} - 2\right) = 2\log\left(\frac{56}{2} - 2\right) = 2.829...$ so a net gain of approximately 3 decimal places will be seen for each additional folded fraction used. Working this through gives

$$r = 4 - \cfrac{8}{31 - \cfrac{1}{30 - \cfrac{1}{30 - \cfrac{1}{30 - \cfrac{1}{30 - \cfrac{1}{30 - \ldots}}}}}}$$

Approximations are as follows

1. 4
2. 3.741935484… correct to 3 decimal places
3. 3.741657696… correct to 6 decimal places
4. 3.741657387118… correct to 8 decimal places
5. 3.741657386774… correct to 11 decimal places

6. CONSIDERATION OF THE VALUE OF β

6.1 Relationship of β to the Root

If $\beta > \varepsilon\left[(\alpha+1)^n - (\alpha)^n\right]$, where ε is the percentage given in the list below, then $\alpha+1$ should be used instead of α , and the value of β should be adjusted to the relevant negative value.

m/n	ε %
1/3	42.10
2/3	48.88
1/4	39.99
3/4	43.07
1/5	35.48
2/5	34.18
3/5	38.70
4/5	39.81
1/6	31.75
1/7	24.28
1/8	21.45

Here's an example using $\sqrt[3]{65}$, If the initial value of α is taken as 4, β will be 1. We check the validity of using this estimate by using the above formula which gives the following question

Is $\beta > 0.421\left[(\alpha+1)^n - (\alpha)^n\right]$?

Is $1 > 0.421[125-64]$?

Is $1 > 25.68$?

The answer is no it isn't, so taking a value of 4 for α will tend towards the actual root more quickly than taking a value of $(\alpha+1)$ or 5.

Here's another example using the cube root of 100.

Giving α an initial value of 4 means $\beta = 100 - 4^3 = 36$. Is it better to use 5 instead of 4? We continue as follows

Is $\beta > 0.421\left[(\alpha+1)^3 - (\alpha)^3\right]$?

Is $36 > 0.421[125-64]$?

Is $36 > 25.68$?

The answer is yes it is, so taking a value of 5 for α will tend towards the actual root more quickly than taking a value of 4.

The corresponding value of β is then $100 - 5^3 = 100 - 125 = -25$

Working through these two options in parallel gives the following results

Approximation	$\alpha = 4, \beta = 36$		$\alpha = 5, \beta = -25$	
1	4		5	
2	4.63157…	1dp	4.642857…	2dp
3	4.64146…	3dp	4.6415929…	4dp
4	4.6415872…	5dp	4.6415888465…	7dp
5	4.6415888135…	7dp	4.641588833653…	10dp
6	4.6415888333…	9dp	4.6415888336129…	12dp

Compared to the actual value of 4.6415888336127…

Whilst both results tend towards the actual root, selecting 5 instead of 4 gives an improvement of almost 1 decimal place at each additional step.

Taken to a higher degree using an initial estimate of 21 for the square root of 1000, which is actually much closer to 31, subsequent estimates will converge to beyond a calculator in about 10 steps.

Approximation	$\alpha = 21, \beta = 559$		$\alpha = 31, \beta = 39$	
1	21		31	
2	31.106758…	0dp	31.622776595531…	6dp
3	31.601582945…	1dp	31.622776601683…	12dp
4	31.6219126805774…	2dp		
5	31.6227413963592…	3dp		
6	31.6227751670629…	5dp		
7	31.6227765432229…	6dp		
8	31.6227765993015…	6dp		
9	31.6227766015867…	9dp		

7. SUMMARY

If you've managed to get this far it's for one of two reasons. Either you have read through in detail and have crashed into this section as it was on your way or you've dropped in, missing a few if not all previous parts, expecting to have an overview of everything on one page.

Either way, although I'm somewhat disappointed in those of you that fall into the latter category, thanks for your interest in a subject that has been dwelling in the back of my mind for quite a while.

Thanks must go to Ron Doerfler for his interest and encouragement in my early efforts towards the final results. I received an email from Ron on the 20th August asking if the method I was exploring could be expanded to include fractional roots. On a drive home from a friend's house late on the 21st August, just coming up to midnight, the idea of how to extract fractional roots began to crystalise. On arriving home I jotted down the idea and tested some numbers, the results of which were very encouraging and led me to the satisfyingly tidy general solution in this document. So indirectly I owe a debt of thanks to Alan Giddens, who I'm sure has no interest in anything mathematical whatsoever, and whom I visited that evening. Had I not, the chain of subsequent events may have been different and no solution ever reached.

I'll always continue using mathematics for recreation, exploring it in ways that are of interest to me, I don't think anything could stop that.

Manny Sardina
Surrey, England
5th September 2007

Footnote: the initial general method is in 3.3h, the general folded method can be found in 4.2e and the subsequent reduction for square roots in 5.1b.