Solution to *The Sensational Mentalist*

A Mathematical Puzzle by Ron Doerfler [1]
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July 12, 2007

The solution to the puzzle is presented here, first as a list of answers and then as a solution with detailed explanations.

Answer List

1. Maximum possible N = 999989999989
2. Minimum possible N = 000089000089
3. List of possible sums of digits of N is (34,52,70,88,106)
4. List of possible sums of odd-place digits of N is (18,38,16,36,34,54)
5. List of possible sums of even-place digits of N is (14,16,34,36,52,54)
6. N = 899999998999

Detailed Solution

There are many ways to attack this problem. The following detailed solution is simply to show that the problem can be solved.

For simplicity, we will use the notation $N \bmod x$ to denote the remainder when N is divided by x.

If we were to take any number at all, say 182175, and express it in multiples of 100, we would have

$$182175 = 18 \times 100 \times 100 + 21 \times 100 + 75$$

Now we can express each 100 as (99 + 1), and if we were to divide 182175 by 99, we would remove all terms multiplied by 99, leaving

$$182175 \bmod 99 = 18 + 21 + 75$$

Therefore, if we know the remainder when any number is divided by 99, we know that the sum of the pairs of digits in the number must have the same remainder when divided by 99. Let us define for this puzzle,

$$N = abcdefghijkl$$

where the digits g, i and k are the same as the digits a, c and e (although possibly mixed up), and the digits h, j and l are the same as the digits b, d and f (although possibly mixed up). Then $(ab + cd + ef + gh + ij + kl)$ equals a positive number less than $(6 \times 99) = 594$ that leaves a remainder of 79 when divided by 99. Possible values are 79, 178, 277, 376, 475, and 574, but since the sum must be even, we can reduce this to 178, 376, and 574.

[1] The puzzle *The Sensational Mentalist* can be found at *http://www.myreckonings.com/Dead_Reckoning/Online/SensationalMentalist.pdf*

Question 1

Obviously, 999999999999 is divisible by 99, because there are 6 pairs of 99's. If we want N mod 99 = 79, we want to change a pair that reduces this maximum number the least. Remembering that we add pairs of digits, we can lower the least significant pair of each half of N (which meets the constraints on N), and by lowering these pairs by 10 apiece, we reduce the total remainder by 20, so we end up with a net remainder of 79. Therefore, the maximum value that N can be is

$$N = 999989999989$$

Question 2

Obviously, 000000000000 is divisible by 99 since it leaves a remainder of 0. Therefore, we want to change pairs of digits in each half of N to get 79, or (99 + 79), or so forth when the pairs are summed. Since we have to change digits in both halves, the sum must be even, the lowest of this type being (99 + 79) = 178. Changing the least significant pair of each half of N, the minimum value that N can be is

$$N = 000089000089$$

Question 3

As mentioned in the prologue to the problem, the nines test (or digit-sum test, or "casting out nines") can be used here. If you sum the digits of any number, the sum must have the same remainder when divided by 9 as the original number had. This can be seen by expressing the original number in multiples of 10 and replacing each 10 by (9 + 1).

Since 99 is a multiple of 9 and we know that N mod 99 = 79, then we can divide 79 by 9 to find the remainder N mod 9 = 7. Therefore, the sum of the digits of N must also leave a remainder of 7 when divided by 9. Since the last 6 digits of N are a mixed-up version of the first 6, we can just double the sum of the first 6. Also, since the maximum sum must be no greater than 12 × 9 = 108, then

$$\begin{aligned}\text{Sum of digits of } N &= 2(a+b+c+d+e+f)\\ &= 7, 16, 25, 34, 43, 52, 61, 70, 79, 88, 97, 106\end{aligned}$$

But the sum must be even, since we are multiplying by 2, so

$$\text{Sum of digits of } N = 16, 34, 52, 70, 88, 106$$

Now more discerning solvers will note that we have really only used a fraction of the information given. We have used N mod 9, but we are given the more selective information, N mod 99 = 79. We can use the rest of the information to narrow our list even more. In particular, we can divide 79 by 11 to find that N mod 11 = 2. If we express a number in multiples of 10 and replace each 10 by (11 − 1), we see that the difference between the sum of the odd-place digits and the sum of the even-place digits must have the same remainder when divided by 11 as the original number. This difference can range from $-(6\times 9)$ to $+(6\times 9)$, or −54 to 54.

$$2(b+d+f) - 2(a+c+e) = -53, -42, -31, -20, -9, 2, 13, 24, 35, 46$$

But the difference must be even, so

$$2(b+d+f) - 2(a+c+e) = -42, -20, 2, 24, 46$$

Now if we take the possible sums of digits and the possible differences of odd-place and even-place digits, and divide both by two and add them, we find

$$\begin{array}{lll} & (b+d+f)+(a+c+e) & = 8,17,26,35,44,53 \\ + & (b+d+f)-(a+c+e) & = -21,-10,1,12,23 \\ \hline & 2(b+d+f) & = 18,36,54,20,38,40,16,34,14,32 \end{array}$$

where we kept only even results between 0 and $6 \times 9 = 54$.

But earlier we found that the sum of the pairs of digits in N could only be 178, 376, or 574, so $2(b+d+f)$, which is the sum of the units digits of each pair, must end in 8, 6 or 4. Therefore, the possible sums of the odd-place digits of N are:

$$2(b+d+f) = 18,36,54,38,16,34,14$$

Now let's sort these out to find the sums of the even-place digits that are possible in N. These are the tens digits in summing the pairs of digits in N, so we subtract the tens digit in each odd-place sum from the tens in the possible sums of pairs. We then add the sums of odd-place digits to those of even-place digits to find the possible total sums of digits of N:

Sum of Pairs	*Sum of Odd-Place Digits*	*Sum of Even-Place Digits*	*Sum of All Digits*
178	18	16	34
	38	14	52
376	16	36	52
	36	34	70
574	14	56	70
	34	54	88
	54	52	106

Possible Sums of Digits of N.

The fifth row is impossible, since the sum of even-place digits cannot be greater than $6 \times 9 = 54$. After using all available information, we find that we have eliminated a sum of 16 from our earlier list of sums of all digits of N. Therefore, we end up with a minimum list of:

$$\text{Sum of digits of } N = (34,52,70,88,106)$$

Question 4

From the above table, remembering to disallow the fifth row,

$$\text{Sum of odd-place digits of } N = (18,38,16,36,34,54)$$

Question 5

From the above table, remembering to disallow the fifth row,

$$\text{Sum of odd-place digits of } N = (18,38,16,36,34,54)$$

Question 6

Jonathon the Tremendous knew that Charles now had enough information to deduce N. Since there are an incredible number of possible N values, the Sum of Even-Place Digits must have narrowed the possible digits down incredibly. The only sum that would do that is 52, which corresponds in the table to an odd-place sum of 54, meaning that all odd-place digits are 9. The even-place digits must consist of four 9's and two 8's to total 52 and have every digit duplicated. From the previous question, the lowest digit in N, which we now know is 8, is located at the 12^{th} and 4^{th} position from the right, so

$$N = 899999998999$$