Mental Square Root Strategies

Joe.Maruszewski@gmail.com

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1 Introduction

This note gives some possible options when using the algorithm decribed in [1] for the calculation of square roots.

2 General Square Root Algorithm

When calculating the square root of a number N using groups of p digits, the algorithm is started by obtaining the initial group x_0 that is the largest p-digit integer for which $x_0^2 < N$. Here we assume that the decimal place in N has been shifted p digits at a time so that N is a $2p$-digit or $(2p-1)$-digit number. The initial group is used to compute the initial remainder r_0:

$$r_0 = 10^p \left(N - a_0^2\right)/2 \qquad (1)$$

The following digit groups can be obtained by using:

$$a_i = r_{i-1}/a_0 \qquad (2)$$

$$r_i = 10^p \left(r_{i-1} - a_i a_0\right) - \frac{1}{2}\sum_{j=1}^{i} a_j a_{i-j+1} \qquad (3)$$

Using the algorithm as presented above will often result in subsequent a_i groups that increase in magnitude. For example,for $N = 6$ with $p = 1$,

N = 6, p = 1		
i	x_i	r_i
0	2	10.0
1	5	-12.5
2	-6	25.0
3	12	-68.0
4	-34	242.0
5	121	-881.0
6	-440	3324.0

To avoid this, in [1] it is suggested to try to keep the magnitudes of 2-digit groups less than 50. Here we suggest an alternative of selecting a_i such that the new remainder will be positive. When doing mental calculations, we have three options for implementing this strategy. The result will be the same but some may find one method easier than the others, depending on their memory and calculation skills.

2.1 Recompute the Remainder

We can be sure to remember the previous remainder r_{i-1} and use it to get get a tentative next digit group a_i from equation 2 above. If the resulting next remainder is negative, we can use the memorized previous remainder and decrement the tentative a_i by one and try again.

Algorithm 1
Choose a_0.
Compute $r_0 = 10^p \left(N - a_0^2\right)/2$
for $i = 1, 2, 3, ...$
  $a_i = r_{i-1}/a_0$
  $r_i = 10^p \left(r_{i-1} - a_i a_0\right) - \frac{1}{2}\sum_{j=1}^{i} a_j a_{i-j+1}$
  while $r_i < 0$
    $a_i = a_i - 1$
    $r_i = 10^p \left(r_{i-1} - a_i a_0\right) - \frac{1}{2}\sum_{j=1}^{i} a_j a_{i-j+1}$

2.2 Adjust the Remainder

Another choice is to notice that everytime we reduce a_i by one, we increase the residual by $10^p a_0 + a_1 - \frac{1}{2}$ if $i = 1$, and by $10^p a_0 + a_1$ otherwise. This leads to the following algorithm.

Algorithm 2
Choose a_0.
Compute $r_0 = 10^p \left(N - a_0^2\right)/2$
for $i = 1, 2, 3, ...$
  $a_i = r_{i-1}/a_0$
  $r_i = 10^p \left(r_{i-1} - a_i a_0\right) - \frac{1}{2}\sum_{j=1}^{i} a_j a_{i-j+1}$
  while $r_i < 0$
    if $i = 1$
      $r_i = r_i - \frac{1}{2}$
    $r_i = r_i + 10^p a_0 + a_1$
    $a_i = a_i - 1$

2.3 Use $2p$-Digit Division

Finally, repeating equation 3,

$$r_i = 10^p \left(r_{i-1} - a_i a_0\right) - \frac{1}{2}\sum_{j=1}^{i} a_j a_{i-j+1}$$

we see that for $i > 1$,

$$r_i = 10^p r_{i-1} - \left(10^p a_0 - a_1\right) a_i - \frac{1}{2}\sum_{j=2}^{i-1} a_j a_{i-j+1} \tag{4}$$

Since we want to ensure that $r_i > 0$,

$$10^p r_{i-1} - \left(10^p a_0 - a_1\right) a_i - \frac{1}{2}\sum_{j=2}^{i-1} a_j a_{i-j+1} > 0 \tag{5}$$

Solving for a_i,

$$a_i < \frac{10^p r_{i-1} - \frac{1}{2}\sum_{j=2}^{i-1} a_j a_{i-j+1}}{10^p a_0 - a_1} \tag{6}$$

So if we are able to do mental division by $2p$ digits, we can directly compute the next a_i for $i > 1$. We still have to use one of the two above algorithms to compute a_1.

The following sections give the details of the calculations when using the strategy of keeping the remainder positive. Here the roots of 3, 6, 12, 17, 39, 51, 80, 82, and 99 are computed using 1-, 2-, and 3-digit division. Note that for 2-digit divisions the decimal place is shifted so that we start with a three or four digit number, and when 3-digit divisions are used the decimal place is shifted so that we start with a five or six digit number.

3 Examples with 1-Digit Division

N = 3, p = 1		
i	x_i	r_i
0	1	10.0
1	7	5.5
2	3	4.0
3	2	1.5
4	0	9.0
5	5	3.0
6	0	15.0
7	8	4.0
8	0	16.0
9	7	12.5
10	6	2.0
11	-4	16.0
12	9	7.0
13	-1	1.0
14	-3	16.0

N = 6, p = 1		
i	x_i	r_i
0	2	10.0
1	4	12.0
2	4	24.0
3	9	16.0
4	5	4.0
5	-1	3.5
6	-1	18.0
7	7	12.5
8	4	15.0
9	3	3.5
10	-3	23.0
11	8	9.5
12	3	14.0
13	2	5.5
14	-3	24.0

N = 12, p = 1		
i	x_i	r_i
0	3	15.0
1	4	22.0
2	6	16.0
3	4	6.0
4	1	2.0
5	0	6.0
6	1	22.0
7	6	9.5
8	1	21.0
9	5	9.0
10	1	16.0
11	3	30.5
12	7	34.0
13	7	28.0
14	5	26.0

N = 17, p = 1		
i	x_i	r_i
0	4	5.0
1	1	9.5
2	2	13.0
3	3	5.0
4	1	3.0
5	0	23.5
6	5	27.0
7	6	13.5
8	2	26.0
9	5	28.0
10	6	12.0
11	1	37.5
12	7	33.0
13	6	33.0
14	6	13.0

N = 39, p = 1		
i	x_i	r_i
0	6	15.0
1	2	28.0
2	4	32.0
3	5	2.0
4	-1	62.0
5	9	53.5
6	8	8.0
7	0	2.5
8	-1	56.0
9	8	35.5
10	4	8.0
11	-1	53.0
12	8	35.0
13	4	15.0
14	-1	55.0

N = 51, p = 1		
i	x_i	r_i
0	7	10.0
1	1	29.5
2	4	11.0
3	1	31.0
4	4	22.0
5	2	61.5
6	8	35.0
7	4	24.0
8	2	66.0
9	8	46.0
10	5	39.0
11	4	30.0
12	2	69.0
13	8	50.0
14	5	13.0

N = 80, p = 1		
i	x_i	r_i
0	8	80.0
1	9	39.5
2	4	39.0
3	4	26.0
4	2	66.0
5	7	21.0
6	1	85.0
7	9	15.0
8	1	7.0
9	0	3.5
10	0	6.0
11	-1	83.5
12	9	22.0
13	1	57.5
14	5	83.0

N = 82, p = 1		
i	x_i	r_i
0	9	5.0
1	0	50.0
2	5	50.0
3	5	37.5
4	3	80.0
5	8	52.5
6	5	20.0
7	1	40.5
8	3	81.0
9	8	23.0
10	1	42.0
11	3	75.5
12	7	52.0
13	4	27.5
14	1	70.0

N = 99, p = 1		
i	x_i	r_i
0	9	90.0
1	9	49.5
2	4	99.0
3	9	91.0
4	8	82.0
5	7	54.5
6	4	49.0
7	3	82.0
8	7	23.0
9	1	19.5
10	0	76.0
11	6	72.0
12	6	33.0
13	2	14.5
14	0	10.0

4 Examples with 2-Digit Division

N = 300, p = 2		
i	x_i	r_i
0	17	550.0
1	32	88.0
2	5	140.0
3	8	131.5
4	7	986.0
5	56	1541.0
6	88	1348.0
7	77	523.5
8	29	641.0
9	35	535.0
10	27	862.0
11	44	1190.0
12	63	803.0
13	41	955.5
14	50	1108.0

N = 600, p = 2		
i	x_i	r_i
0	24	1200.0
1	49	1199.5
2	48	2398.0
3	97	1095.0
4	42	1986.0
5	78	857.5
6	31	2013.0
7	78	342.0
8	9	2132.0
9	82	40.0
10	-3	844.0
11	28	1075.5
12	40	1973.0
13	74	1847.0
14	70	1652.0

N = 1200, p = 2		
i	x_i	r_i
0	34	2200.0
1	64	352.0
2	10	560.0
3	16	526.0
4	15	480.0
5	13	2690.0
6	77	1902.0
7	54	2053.5
8	58	2471.0
9	70	1936.5
10	54	3155.0
11	89	1007.5
12	26	2970.0
13	83	160.0
14	1	731.0

N = 1700, p = 2		
i	x_i	r_i
0	41	950.0
1	23	435.5
2	10	2320.0
3	56	1062.0
4	25	2565.0
5	61	3179.0
6	76	2542.0
7	60	2331.5
8	55	4.0
9	-2	975.5
10	21	1771.0
11	41	36.0
12	-2	2395.0
13	55	4076.0
14	97	1777.0

N = 3900, p = 2		
i	x_i	r_i
0	62	2800.0
1	44	6232.0
2	99	5044.0
3	79	6223.5
4	98	2617.0
5	39	5361.5
6	84	51.0
7	-2	1389.0
8	20	3760.0
9	58	3033.5
10	46	5724.0
11	89	2138.0
12	31	1495.0
13	20	6068.0
14	94	87.0

N = 5100, p = 2		
i	x_i	r_i
0	71	2950.0
1	41	3059.5
2	42	6028.0
3	84	2074.0
4	28	3924.0
5	54	2082.0
6	28	3632.0
7	50	46.0
8	-1	5777.0
9	80	20.0
10	-1	2953.0
11	39	7059.0
12	98	942.0
13	11	2743.0
14	36	5302.0

N = 8000, p = 2		
i	x_i	r_i
0	89	3950.0
1	44	2432.0
2	27	1712.0
3	19	899.5
4	9	8941.0
5	99	8220.5
6	91	5302.0
7	58	7069.5
8	78	5132.0
9	56	3408.5
10	36	6341.0
11	69	4343.5
12	46	6875.0
13	74	8443.0
14	92	4775.0

N = 8200, p = 2		
i	x_i	r_i
0	90	5000.0
1	55	3487.5
2	38	4660.0
3	51	3473.0
4	38	1272.0
5	13	6740.5
6	74	1548.0
7	16	5723.0
8	62	6014.0
9	65	6756.5
10	73	7433.0
11	80	7509.0
12	81	6222.0
13	67	22.0
14	-2	3842.0

N = 9900, p = 2		
i	x_i	r_i
0	99	4950.0
1	49	8699.5
2	87	4387.0
3	43	7108.5
4	71	730.0
5	6	6204.5
6	62	37.0
7	-1	5476.5
8	54	7399.0
9	73	4548.0
10	44	8070.0
11	79	8312.0
12	82	1172.0
13	10	268.5
14	1	2160.0

5 Examples with 3-Digit Division

N = 30000, p = 3		
i	x_i	r_i
0	173	35500.0
1	205	13987.5
2	80	131100.0
3	756	153820.0
4	887	126685.0
5	729	61827.0
6	352	129948.0
7	744	110811.5
8	634	27275.0
9	150	103121.5
10	587	42325.0
11	236	121574.0
12	694	49954.0
13	280	92709.0
14	525	67560.0

N = 60000, p = 3		
i	x_i	r_i
0	244	232000.0
1	948	238648.0
2	974	68648.0
3	278	78118.0
4	317	198712.0
5	809	201668.0
6	819	179496.0
7	728	101003.5
8	407	116457.0
9	470	145774.5
10	589	35855.0
11	139	49530.5
12	196	147079.0
13	594	185074.0
14	748	18062.0

N = 120000, p = 3		
i	x_i	r_i
0	346	142000.0
1	410	55950.0
2	161	177990.0
3	513	268709.5
4	775	159157.0
5	458	244860.5
6	705	170137.0
7	489	93738.5
8	268	105276.0
9	301	61328.0
10	174	164850.0
11	473	135468.5
12	388	195241.0
13	561	18664.5
14	50	265111.0

N = 170000, p = 3		
i	x_i	r_i
0	412	128000.0
1	310	231950.0
2	562	231780.0
3	561	316168.0
4	766	23258.0
5	54	405407.5
6	982	59006.0
7	140	407044.0
8	985	247704.0
9	597	169150.0
10	407	291463.0
11	702	213497.0
12	514	298837.0
13	719	382350.0
14	922	223884.0

N = 390000, p = 3		
i	x_i	r_i
0	624	312000.0
1	499	499499.5
2	799	524799.0
3	839	525138.5
4	839	513478.0
5	820	366498.5
6	584	431983.0
7	689	196632.5
8	312	60345.0
9	93	613058.0
10	979	280453.0
11	446	68922.0
12	107	186629.0
13	296	399.5
14	-3	500466.0

N = 510000, p = 3		
i	x_i	r_i
0	714	102000.0
1	142	601918.0
2	842	610436.0
3	854	204250.0
4	285	462.0
5	-1	571514.0
6	799	671994.0
7	939	702145.5
8	981	99499.0
9	136	520064.5
10	726	378306.0
11	527	627883.5
12	876	443330.0
13	617	85282.5
14	115	709260.0

N = 800000, p = 3		
i	x_i	r_i
0	894	382000.0
1	427	170835.5
2	190	894370.0
3	999	819377.0
4	915	786485.0
5	878	505243.5
6	563	600194.0
7	669	419632.5
8	467	442174.0
9	492	458268.0
10	510	443768.0
11	494	159482.5
12	176	223145.0
13	247	309257.5
14	343	757076.0

N = 820000, p = 3		
i	x_i	r_i
0	905	487500.0
1	538	465278.0
2	513	737006.0
3	813	672021.5
4	741	600773.0
5	662	596226.5
6	657	345995.0
7	380	740772.5
8	816	633869.0
9	698	369969.0
10	406	582576.0
11	641	278349.5
12	305	194429.0
13	212	406112.0
14	446	372912.0

N = 990000, p = 3		
i	x_i	r_i
0	994	982000.0
1	987	434915.5
2	437	106181.0
3	106	616893.5
4	619	950225.0
5	954	731281.0
6	734	478030.0
7	479	817764.5
8	821	2520.0
9	1	206058.0
10	206	6478.0
11	5	178394.0
12	178	126854.0
13	126	561519.5
14	563	674277.0

References

[1] R. Doerfler, "Dead Reckoning: Calculating Without Instruments", Gulf Publishing, 1993.