Developments in the Useful Circular Nomogram*

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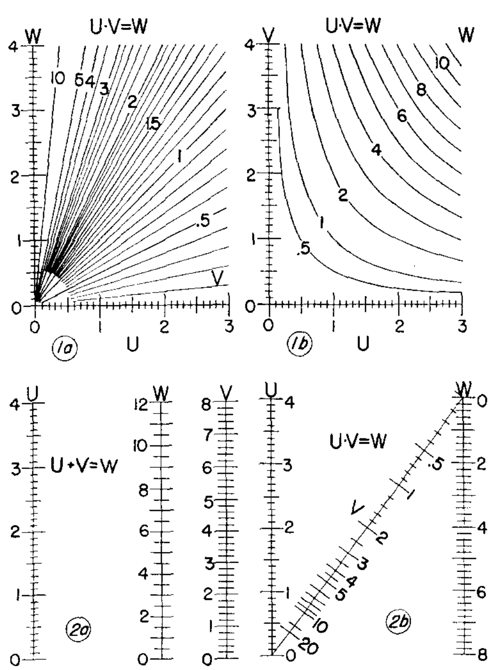
(Received August 30, 1948)

The particular type of alignment diagram for the function $U\cdot V=W$ in which the U and V scales lie on the circumference of the same circle and the W scale along a diameter is developed in detail. The forms of the diagram for which certain intervals on these scales are maximized in their spread are derived and described. Such maximized diagrams, it is found, have certain characteristic properties of symmetry. These developments are illustrated with examples.

BACKGROUND

A NOMOGRAM, or "law in graphical form," is a diagram designed to represent graphically a functional relationship between varying quantities. In particular, a nomogram consists of points, lines, or curves—each calibrated in one of the varying quantities of the relationship. When all the quantities but one have known values in a relationship for which a diagram has been prepared, the value of the unknown one can be found by reference to a point, line, or curve, characteristic of the solution, established by the known values. Extended use of the diagram can also often be made to determine compatible values of more unknowns than one.

The ordinary network chart is such a nomogram.



FIGS. 1a, 1b, 2a, and 2b. Elementary forms of network chart and alignment diagram.

Thus, for the equation $U\cdot V=W$, it might take the form of Fig. 1(a) or Fig. 1(b). On reference to the point P_j, characteristic of the solution established by known values U_j and W_j, the unknown value V_j can be read from the chart. The alignment diagram is likewise such a nomogram. It could take, for instance, either of the well-known forms of Fig. 2(a) or Fig. 2(b). On reference to the line K_j characteristic of the solution established by the known values U_j and W_j, the unknown value V_j can be read from the diagram.

The generalized alignment diagram for the three-variable equation,

$$F(U,\ V,\ W)=0,$$

can be schematically represented in the form of Fig. 3. The U, V, W curves may partake of any definable form. Thus, one of them might be circular (Fig. 4). This type of diagram is not especially rare in the literature. It is clearly distinguished from the type which is of special interest in this article.

THE CIRCULAR NOMOGRAM

The equation $U\cdot V=W$ can also be represented in alignment diagram form by the use of a pair of coincident, circular U and V scales and a third straight-line W scale lying along a diameter of the circle. The values U, V, $W=0$ coincide at one end of the diameter and the values U, V, $W=\infty$, U, V, $W=-\infty$ coincide at the other end (Fig. 5). This diagram, extremely useful for certain purposes, seems to be employed by current English nomographers, but to be practically unknown in this country. In a survey of seventeen hundred nomograms appearing in the technical periodical literature in this country for the years 1925–1947, only one, designed by an Englishman, was found which partook of this form.

Geometrical Relationships

In Fig. 6, let line ABC be a random line of collineation cutting proposed U, V, W scales at points A, C, and B, respectively. Draw lines OA and OC defining peripheral angles θ and ϕ, which vary directly with arcs EA and EC. Construct line

* The circular nomogram has been given a thorough introductory treatment in *Elements of Nomography* by R. D. Douglass and D. P. Adams (McGraw-Hill Book Company, Inc., New York, 1947). The present authors believe that treatment to be the first in American texts.

ED parallel to line AC and draw line DC. Since $DC = AE$,

$$\measuredangle DOC = \measuredangle AOE,$$
$$\measuredangle DOC = \theta,$$
$$\measuredangle OAC = \measuredangle ODC.$$

Triangles OAB and ODC are similar and

$$OB/OA = OC/OD,$$
$$OA = 2a\cos\theta,$$
$$OC = 2a\cos\phi,$$
$$OD = 2a\cos(\phi-\theta).$$

Let OB, the distance from O to the cutting point B along the proposed W scale, be called S_W. It is the scale distance measured from point O out along the diameter to the point of location of any particular value W.

$$S_W = 2a\cos\theta\cos\phi/(\cos\theta\cos\phi+\sin\theta\sin\phi),$$
$$S_W = 2a/(1+\tan\theta\tan\phi).$$

Let the U and V scales be defined upon their circular arcs by the relationships:

$$U = \tan\theta, \quad (1)$$

$$V = \tan\phi, \quad (2)$$

$$S_W = 2a/(1+U\cdot V),$$
$$S_W = 2a/(1+W). \quad (3)$$

Thus, the W scale equation is observed to be identical with that for the diagonal scale for the N-type diagram, Fig. 2(b), when the latter is measured from the $V=\infty$ end.[1] This makes it possible to compound the circular diagram with the N diagram. A completed circular diagram appears in Fig. 7.

DISCUSSION

Range and Scope of the Circular Nomogram

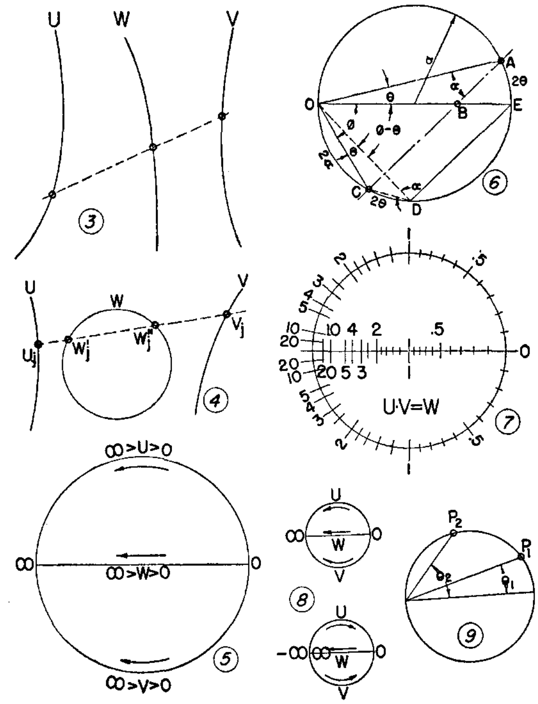
The great advantage of the circular nomogram lies in its closure, which permits infinite ranges of all three variables to be present. This property is in contrast to the diagrams for the same basic formula for multiplication presented in Fig. 2(b). More generally, one can have the two arrangements presented schematically in Fig. 8. Any adaptable portion of such a diagram will be a valid alignment diagram.

Changing the Distribution of Graduations

Figure 7 shows the unit values of all three scales on a vertical line through the center of the circle. Fortunately, this distribution of values along the scale can be varied. Assume

$$P\cdot Q = R,$$
$$mP\cdot nQ = mnR.$$

[1] R. D. Douglass and D. P. Adams, *Elements of Nomography* (McGraw-Hill Book Company, Inc., New York, 1947), p. 100, Note 2.



FIGS. 3–9. Basic characteristics of the circular nomogram.

Let

$$U = mP,$$
$$V = nQ,$$
$$W = mnR;$$

then

$$U\cdot V = W.$$

A diagram in U, V, W can be drawn in the standard form of Fig. 7. However, the diagram graduated in the original variables P, Q, and R will be such that these variables along the vertical line through the center of the circle will be $P_0 = 1/m$, $Q_0 = 1/n$, and $R_0 = 1/mn$. Great freedom of distribution of values is thus achieved. The standard equations 1, 2, and 3 will now be

$$P = \tan\theta/m, \quad (4)$$

$$Q = \tan\phi/n, \quad (5)$$

$$S_R = 2a/(1+mnR). \quad (6)$$

Maximizing Intermediate Scale Intervals

If the lowest values of the more useful ranges of the variables P and Q are greater than zero and the highest values of these ranges are finite, the more useful range of the R scale will not contain $R=0$ or $R=\infty$. The *spread* of this useful range will vary with m and n. A preliminary plan of the diagram may show this more useful range to be embarrassingly small. However, it can be maximized as follows. Let R_1 and R_2 be the lowest and highest values in the more useful range of R. Let this R

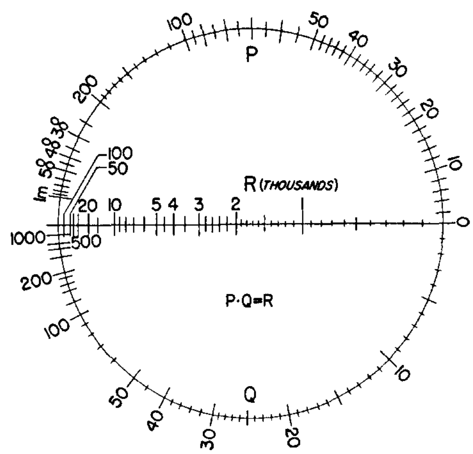


FIG. 10. A simple circular nomogram with maximized ranges.

range be maximized as a problem independent of the other scales. The R scale is characterized by the multiplier $l=mn$:

$$\begin{aligned} S_R &= 2a/(1+lR),\\ S_{R_1} &= 2a/(1+lR_1),\\ S_{R_2} &= 2a/(1+lR_2),\\ T &= S_{R_1}-S_{R_2}. \end{aligned}$$

Then T, the spread of the most used range of R, is to be maximized with respect to the variable l:

$$T=2a\left(\frac{1}{1+lR_1}-\frac{1}{1+lR_2}\right),$$

$$\frac{dT}{dl}=2a\left(\frac{R_2}{(1+lR_2)^2}-\frac{R_1}{(1+lR_1)^2}\right).$$

For a maximum,

$$R_1(1+2lR_2+l^2R_2^2)=R_2(1+2lR_1+l^2R_1^2),$$
$$l=1/(R_1R_2)^{1/2},$$
$$\left.\begin{aligned} S_{R_1} &= 2a/[1+(R_1/R_2)^{1/2}]\\ S_{R_2} &= 2a/[1+(R_2/R_1)^{1/2}]\\ S_R &= 2a/[1+[1/(R_1R_2)^{1/2}]R] \end{aligned}\right\}. \qquad (7)$$

It is interesting to observe that the position of this spread upon the diameter is symmetric with respect to the center of the circle. Thus,

$$\frac{S_{R_1}+S_{R_2}}{2}=\frac{2a}{2}\left(\frac{1}{1+(R_1/R_2)^{1/2}}+\frac{1}{1+(R_2/R_1)^{1/2}}\right)$$
$$=a.$$

Maximizing the circular scale intervals, one has (Fig. 9):

$$\begin{aligned} \theta_1 &= \tan^{-1}mP_1,\\ \theta_2 &= \tan^{-1}mP_2,\\ A &= 2(\theta_2-\theta_1). \end{aligned}$$

Then A is the spread of the most used range of P and is to be maximized with respect to m.

$$dA/dm=2[(P_1/(1+m^2P_1^2))-(P_2/(1+m^2P_2^2))].$$

For a maximum,

$$-P_1-m^2P_1P_2^2+m^2P_2P_1^2+P_2=0,$$
$$m=1/(P_1P_2)^{1/2}.$$

Correspondingly,

$$n=1/(Q_1Q_2)^{1/2}.$$

These ranges are also distributed symmetrically with respect to a vertical line on the circle, for

$$\begin{aligned} \tan\theta_1 &= (P_1/P_2)^{1/2},\\ \tan\theta_2 &= (P_2/P_1)^{1/2},\\ \tan\theta_1 &= \cot\theta_2\\ &= \tan(90°-\theta_2),\\ (\theta_1+\theta_2)/2 &= 45°. \end{aligned}$$

Let the P and Q ranges be maximized independently and the R range be regarded as dependent.

$$\begin{aligned} P_1Q_1 &= R_1,\\ P_2Q_2 &= R_2,\\ l &= mn,\\ l &= 1/(P_1Q_1P_2Q_2)^{1/2},\\ l &= 1/(R_1R_2)^{1/2}. \end{aligned}$$

Thus, the R range is also maximized.

In this way, a circular diagram may be drawn with maximum dispersions for the most useful portions of all three scales. In addition, the maximization process, because of its symmetry with respect to the vertical diameter described above, produces a chart in which the index line (e.g., line ABC in Fig. 6) makes the greatest angle with the dependent scale on the diameter, thus allowing the most accurate measurement of R. Hence, the problem of optimum placement of scales for intermediate ranges of the variables in the circular nomogram achieves a pleasing and classical simplicity.

APPLICATIONS

Illustrative Problem 1

Figure 10. Design a circular nomogram six inches in diameter for the equation $P\cdot Q=R$ where infinite ranges of the variables are desirable but the ranges of P from $P_1=50$ to $P_2=100$, of Q from $Q_1=20$ to $Q_2=30$, and of R from $R_1=1000$ to $R_2=3000$ are subject to especially heavy use.

$$\begin{aligned} m &= 1/(50\cdot 100)^{1/2}=0.01414,\\ n &= 1/(20\cdot 30)^{1/2}=0.04082,\\ l &= 1/(1000\cdot 3000)^{1/2}=0.0005773,\\ \theta &= \tan^{-1}0.01414P,\\ \phi &= \tan^{-1}0.04082Q,\\ S_R &= 6/(1+0.000577R). \end{aligned}$$

Illustrative Problem 2

Figure 11.[2] Design a circular nomogram six inches in diameter for the equation

$$(\log \sin A)(B^3-4)=7\cos^2 C.$$

Place the B scale on the diameter of the circle. All first-quadrant values of the angles A and C are useful but the range of A from $A_2=10$ to $A_1=50$ and of C from $C_1=10$ to $C_2=50$ are heavily used.

Solution: Since the variable B is to be recorded upon the diameter, write the equation in the form

$$(\log \sin A)(1/7\cos^2 C)=1/(B^3-4),$$
$$U'=-\log \sin A,$$
$$V'=1/7\cos^2 C,$$
$$W'=-1/(B^3-4),$$
$$U_1'=0.1158,$$
$$U_2'=0.7603,$$
$$m=3.37,$$
$$V_1'=0.1473,$$
$$V_2'=0.3457,$$
$$n=4.44,$$
$$l=14.95.$$

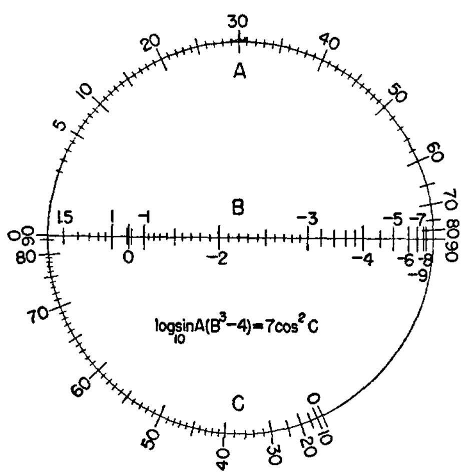


FIG. 11. An advanced circular nomogram with maximized ranges.

Then useful scale equations[3] for plotting this diagram are:

$$S_A=-6(3.37\log \sin A) \quad\text{or}\quad S_A=-6/3.37\log \sin A;$$
$$S_C=6(4.44/7\cos^2 C) \quad\text{or}\quad S_C=6(7\cos^2 C/4.44);$$
$$S_B=6/[1-(14.95/(B^3-4))].$$

Illustrative Problem 3

Figures 12 and 13.[4] Design a nomogram for the

$$s=r_s\cot\cos^{-1}(\cos\bar{\mu}-d^*),$$

equation where $\bar{\mu}$ ranges from 0° to 35°, d^* ranges from 0 to 1.0, s ranges from 10 to 70, r_s ranges from 15 to 40, in steps of 5. s is the dependent variable in practice. Experience indicates that it has a most heavily used interval from $s=25$ to $s=60$.

Analysis

The presence of four variables, three of them independent, precludes the possibility of a single alignment form for the diagram and requires a compounding of diagrams. Let

$$U=\cot\cos^{-1}(\cos\bar{\mu}-d^*).$$

Then,

$$s=r_s U.$$

Since U is given by $\tan\theta$ in the preceding theory, the presence of the cotangent function militates strongly in favor of the circular nomogram.

Let α be the complementary angle to θ, Fig. 12(a). Then $U=\tan\theta$ is properly recorded as $U=\cot\alpha$. Let

$$\alpha=\cos^{-1}(\cos\bar{\mu}-d^*),$$
$$\cos\alpha=\cos\bar{\mu}-d^*=x/2a,$$
$$x=2a(\cos\bar{\mu}-d^*).$$

The problem of the last equation is easily solved by a diagram of the form of Fig. 12(b). Let the x scale be tangent to the circle at the vertex A of angle α, have zero value there and be $2a$ units long. Let the $\bar{\mu}$-scale be located pleasingly upon the vertical center line of the circle. Let the values $\bar{\mu}=0$ and $d^*=0$ be on a level with $x=2$. Distances along the x scale can be conveniently transferred to the U scale by concentric arcs about point A, particularly with the aid of corresponding arbitrary graduations on the x and U scales (Fig. 12(c)).

Derivation of the Scale Equations

s scale: To maximize the most favored s interval,

$$l=1/(25\cdot 60)^{\frac{1}{2}}=0.0258;$$
$$S_S=10/(1+0.0258s)$$

for a circle of ten inches diameter ($a=5$).

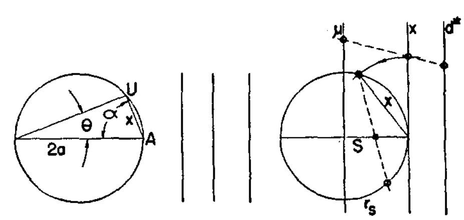


FIG. 12. Breakdown of parts for nomogram for camera.

[2] Compare with Fig. 88 in Douglass and Adams, reference 1, p. 156, where no interval has been maximized.

[3] For derivation and application of scale equations for the circular nomogram, see Douglass and Adams, reference 1, Chapter XVII.

[4] H. T. Evans, Jr., S. G. Tilden, Jr., and D. P. Adams. "New techniques applied to the Buerger precession camera for x-ray diffraction studies," Rev. Sci. Inst. 20, 155 (1949).

In practice, this diagram has been found to be practically indispensable to the efficient use of the equipment whose behavior is represented by this equation.

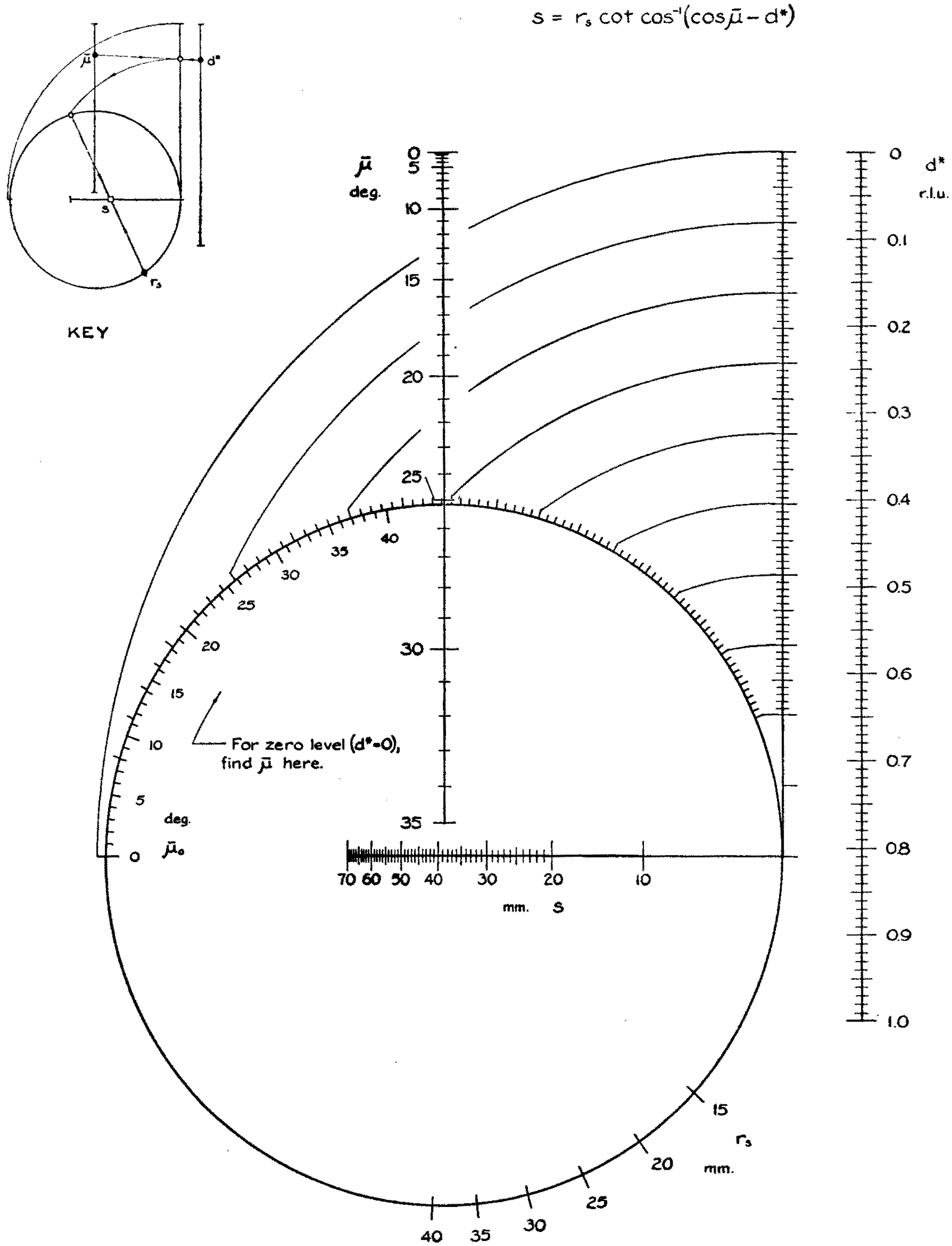


FIG. 13. Final Buerger precession camera nomogram.

r_s scale: The original equation now reads

$$ls = (nr_s)[m \cot \cos^{-1}(\cos\bar{\mu} - d^*)].$$

The device employed requires that

$$\cot\alpha = m \cot \cos^{-1}(\cos\bar{\mu} - d^*).$$

Hence $m=1$, $n=l$ in this scheme and neither of the circular scales can be varied for maximization. The value of r_s on the vertical center line will be

$$r_s = 1/l = 38.73.$$

The resulting r_s scale does lie in part to the left of the center line and has a satisfactory spread without maximization. A convenient scale equation for drafting the r_s scale will be

$$S_{r_s} = 10 r_s.$$

x scale: The scale factor of the x scale is unity. The sum of the scale factors of the d^* and $\bar{\mu}$ scales will equal their product and the distances of these scales from the x scale will be proportional to these scale factors.[5] Of these four quantities, only the distance of the $\bar{\mu}$ scale has been fixed (at five inches) by the present design.

$\bar{\mu}$ scale and d^* scale: The equations for these scales, reduced to scale measurement upward from the horizontal center line of the circle, can accordingly be

$$S_{\bar{\mu}} = 52.6(\cos\bar{\mu} - 0.810),$$
$$S_{d^*} = 12.34(0.810 - d^*).$$

The d^* scale is 1.17 inches from the x scale.

[5] For a description of the derivation of scale factors and scale equations for the parallel-line diagram, see Douglass and Adams, reference 1, Chapter IX.

THE REVIEW OF SCIENTIFIC INSTRUMENTS VOLUME 20, NUMBER 3 MARCH, 1949

New Techniques Applied to the Buerger Precession Camera for X-Ray Diffraction Studies

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The usefulness of the Buerger precession method for single crystal x-ray diffraction studies is increased by the following techniques: (1) The use of a nomogram to facilitate the setting and operation of the instrument; (2) the simplification of orientation and lattice measurement techniques; and (3) the application of the Dawton photographic method to precession photographs for obtaining quantitative intensity data, together with a graphical method for determining Lorentz and polarization corrections.

In 1944, a new method of obtaining x-ray diffraction data from single crystals was described by Buerger.[1] By use of a camera of novel design, aptly called the "precession" camera, layers of the reciprocal lattice were photographed directly without distortion. This instrument has since been made available commercially and has earned favor in several laboratories.[2] It seems apparent, however, that the full possibilities of this instrument have not been generally realized. Therefore, some of the techniques developed in this laboratory in connection with the precession method will perhaps be of interest.

INSTRUMENT SETTINGS

In preparing the instrument for a run, five separate adjustments must be made for each photograph, defined as follows: (1) F, the film-to-crystal distance; (2) $d^{*\prime}$, an adjustment of the film position for proper registration of the upper levels of the lattice; (3) $\bar{\mu}$, the angle between the direct beam and the normal to the plane being photographed; (4) r_s, the radius of the layer-line screen; (5) s, the screen-to-crystal distance.

F is an independent adjustment determined by the desired magnification of the lattice image, and $d^{*\prime}$ is simply the product of F and d^* the reciprocal lattice spacing normal to the planes being photographed. $\bar{\mu}$, r_s, s, and d^* are, on the other hand, related in a rather complex way, as shown by Buerger, thus: $s = r_s \cot \cos^{-1}(\cos\bar{\mu} - d^*)$.

s is usually the last adjustment made and, hence, may be regarded as the dependent variable of the equation. In his monograph, Buerger gives a chart permitting rapid determination of s over a range of d^* (0 to 1.0 reciprocal lattice units) for two values of r_s (15 and 30 mm) and a fixed value of $\bar{\mu}$ (20°). Although these restrictions on r_s and $\bar{\mu}$ are not inconvenient in a majority of cases, there are many times when a greater flexibility of these variables is desirable.

It is possible to present the behavior of the variables in the equation in an alignment diagram which will give the value of s for all useful values of r_s, $\bar{\mu}$, and d^*. By using this diagram, reproduced in Fig. 1, it is an easy matter quickly and accurately to arrive at an optimum value for each of the settings involved. The key to the diagram shows the general scheme to be the employment of two elementary types of alignment diagrams. The first of these is the familiar three-parallel-line diagram; the second is the less familiar circular nomogram. The further adaptation of the latter to the special needs of the

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[1] M. J. Buerger, "The Photography of the Reciprocal Lattice," ASXRED Monograph No. 1 (1944).
[2] See for example, G. L. Clark and H. Kao, J. Am. Chem. Soc. 70, 2151 (1948).