The 13th Root of a 100-Digit Number

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Mental calculators of note (so-called lightning calculators) developed areas of expertise in performing calculations that seem astonishing, even unbelievable, to the rest of us. One such specialty is calculating the 8-digit root of a 13th power of 100 digits. Achieving record times historically required massive memorization and calculating speed, racing through a procedure that remains a mystery to most people. This paper first provides a historical overview of the extraction of 13th roots, including the methods used by a few mental calculators, methods that largely rely on a mix of intensive mental calculation and large-scale rote memorization. It demonstrates the creativity and drive of these marvelous people.

This background information demonstrates the difficulty of the problem and the efforts of calculators to master it. As an alternative, we also propose a new method for 13th roots that is relatively easy to learn, one that makes this feat feasible for those of us with basic mental math capabilities and a desire to do something amazing.

Why 13th Roots?

It is generally known that the difficulty of mentally solving for integer roots depends on the number of digits in the root rather than the number of digits in the power. So why did the 13th root of a 100-digit number become the standard?

First, the appeal of a prime power must be acknowledged. Square root extraction is another popular category in itself. Composite roots such as the 4th root or 12th root can be calculated as a sequence of roots of their factors (two square roots to get a 4th root, followed by a cube root to get a 12th root). In fact, for a given number of digits in the root, even-numbered roots are more difficult because the final digit of an odd root can be found from the final digit of the power. It turns out that

This paper can be found at http://www.myreckonings.com/wordpress/wp-content/uploads/13thRoots/13thRoots.pdf
orders of powers that are one more than a multiple of 4 (such as the 13th power) have a root with the same final digit, while orders of powers that are one less than a multiple of 4 (such as a cube) have a root with a unique final digit relative to the final digit of the power. So here we have the first clue: the final digit of a 13th root will be the same as that of the power.

Second, 100 is an impressive round number of digits, and this produces a 13th root consisting of 8 digits. This number of digits proves to be non-trivial while not beyond the capabilities of the best mental calculators. If there were only 3 digits in the root of an odd power, the problem is easy. The final digit is found from the rules just described. The first digit can be inferred from memorized or estimated ranges of the powers. The properties of modular arithmetic can reveal the middle digit; here the root and power are replaced with remainders after division by an integer while still retaining the 13th power relationship, and we can deduce the missing digit.

For example, consider the 3-digit root of an odd power such as 24137569. Since \(200^3 = 8000000\) and \(300^3 = 27000000\), we know the root has a 2 as the first digit. Since 3 is one less than a multiple of 4, the final digit of the root will not necessarily equal that of the power, but there will be a unique mapping—in this case \(9^3\) ends in 9 so the last digit is in fact 9 and we are left with \(2b9\), where \(b\) is unknown. The remainder when a number \(N\) is divided by 11 (or \(N \mod 11\)) can be found by subtracting the odd-place digits from the even place digits and repeating, adding 11’s until the result lies between 0 and 10. Here \(24137569 \mod 11 = (9 + 5 + 3 + 4) - (6 + 7 + 1 + 2) = 5\). Now \((2b9)^3\) mod 11 must equal 5 as well. So \([\left(9 + 2\right) - b]^3\) mod 11 = \((11 - b)^3\) mod 11 = 5. As noted above, for cube roots there is a unique mapping between \(n\) and \(n^3 \mod 11\) and this mapping would be memorized by the mental calculator. Here we can see that \(3^3 \mod 11 = 5\) since \(27 = 22 + 5\), so \(11 - b = 3\) or \(b = 8\). So the cube root of 24137569 is 289.

For the 13th root of a 100-digit number, the first digit is always 4, and we know the last digit is the same as the power. But the 13th root has 8 digits and 7992563 possibilities so there is much more ambiguity, even when the performer has memorized long tables of 2-digit or 3-digit sets of beginning and ending digits. For this reason, the Guinness Book of World Records created the category of 13th root extraction of 100-digit numbers, recording in the eleventh edition of 1972 a time of 23 minutes by Herbert B. de Grote of Mexico.

**Historical Methods**

In the years since de Grote’s initial record, great efforts have been taken to solve 13th roots, and as a result the times required for it have steadily decreased. Here we will discuss three of the major players in this field: Wim Klein, Gert Mittring and Alexis Lemaire.
Wim Klein of the Netherlands, a lightning calculator who worked at CERN, bested de Grote with a time of 5 minutes and then proceeded to lower his time even further. He eventually attained a record time in 1981 of 1 minute 28 seconds to calculate the 13th root of

8800844340489299575219015772236417859411720052615
65487280650870412023307854274990144578442271602817

He found the answer to be 48757377.

How did he calculate these roots? Klein used logarithms to find the first five digits, and then used his knowledge of 13th root endings and modular arithmetic to deduce the last three digits. To find the logarithms he would factor the initial digits and add up those 5-digit logarithms from memory, interpolating between values as needed for offsets. Then he would divide by 13 and use the reverse process to find the antilogarithm, the number whose logarithm would be that value. This would be the initial five digits of the root. It is far easier said than done.

Smith [1983] discussed with Klein his method for finding the 13th root of

14762420839370760705665953772022217870318956930659
27236796230563061507768203333609354957218480390144

Here is Smith’s account of Klein’s procedure:

The first five digits of the root are fixed through the use of logarithms. Klein has memorized to five places the logs of the integers up to 150; this, coupled with his ability to factor large numbers, allows him to approximate the log of the first five digits of the power, which is usually sufficient to determine the first five digits of the root, though, as he says “the fifth digit is a bit chancy.”

Klein began by factoring 1,476 into 36 times 41 and taking the (decimal) log of each: \(\log 36 = 1.55630\) and \(\log 41 = 1.61278\); adding the mantissas yields 0.16908, but this is, of course, too little. Through various interpolations Klein estimated the mantissa of the log of 147,624 as 0.16925 (it is more nearly 0.16916).

Klein now had an approximation of the log of the 100-digit number above—99.16925. This must be divided by 13 to obtain the log of the 13th root. Since \(99 = 13 \times 7\) with a remainder of 8, to obtain the mantissa of the antilog of the 13th root he divided 8.16925 by 13, which is approximately 0.62840. He estimated the antilog to be about halfway between 4.2 and 4.3 and decided to try 4.25. The result was exact, so the first five digits of the root should be 42500, as indeed they are.

It is now necessary to determine the last three digits of the root. This he does from an examination
of the last three digits of the power. In the case of odd powers, these uniquely determine the last three digits of the root, but in the case of even roots, like this one, this method yields four possibilities; in the case of 144 they are 014, 264, 514, and 764. (The choices always differ by 250.) To select the correct one Klein divides the original number by 13 and retains the remainder. In the case of 13th roots, the root remainder and the power remainder must be the same. The power remainder is 7; only 764 as the final three digits of the root will yield 7 as the remainder. Thus the 13th root is determined to be 42,500,764.

As we will see, there are multiple endings possible when the 13th power ends in 2, 4, 6, or 8, so these are not going to be record attempts. In fact, the above account appears to be unique; other accounts of 13th roots are limited to odd final digits, and the method described in this paper is also limited to odd final digits.

Here is Klein’s account for another 13th power, also from Smith:

75185285487713563581947553291145079861723813162341
53935861550997297991815299022662358976308065985831

The first five digits of the power are 75185, which is nearly 7519, and 7519 is 73 times 103. The mantissa of the log of 73 is 0.86332 and that of 103 is 0.01284. Their sum is 0.87616. Dividing 8.87616 by 13 yields 0.68278. This falls between the mantissas of the logs of 48 and 49, but is much closer to 48. Since 481 is 13 (mantissa 0.11394) times 37 (mantissa 0.56820), the mantissa of its log will be 0.68214; close, but still a bit low; 4,816 can be factored into 16 (mantissa 0.20412) times 7 (mantissa 0.84510) times 43 (mantissa 0.63347). This gives a mantissa of 0.68269. Then 4,818 factors into 66 (mantissa 0.81954) times 73 (mantissa 0.86332), which yields a mantissa of 0.68286. Thus, in the interpolation we want 9/17 of 20 which is about 10\(^{1/2}\). The first five digits of the root should be 48170 (48160 + 10). This, in fact, is correct.

When Klein actually did the calculation he made a minor error (he was looking for the antilog of 0.68277 instead of 0.68278) and first took 48169 for the first five digits of the root. In this case, however, since the root is odd, the last three digits are uniquely fixed—since the power ends in 831, the root must end in 311. Upon dividing the power by 13 Klein got a remainder of 7. But dividing 48,169,311 by 13 gives a remainder of 8. To make these two remainders come into line he changed his solution to 48,170,311, which is correct.

Hope [1985] remarks that “acquaintances of Klein report that during these complex mental calculation tasks, Klein mutters constantly in Dutch while calculating, and a good part of his muttering consists of swearing.”

Klein also worked with higher-order roots. In 1976, he found the 73rd root of a 500-digit number in 2 minutes 43 seconds, a feat recorded in the two photographs of Klein in this paper. In 1983 he found the 73rd root of a 505-digit number in 1 minute 43 seconds. A 73rd root duplicates the final digit of the power just as the 13th root does, since they are both of order \(4k + 1\). No doubt
Klein used the same procedure described above but divided the logarithm by 73 rather than 13. In this case there are only 7 digits with 273696 possibilities, a factor of nearly 30 fewer than the 13th root of a 100-digit number. As we have seen, the mass of digits in the middle of the power mean nothing to the mental calculator, only the starting and ending digits. Klein talked freely of how he would sometimes write down intermediate results, particularly in front of audiences [Smith 1983].

Gert Mittring

These examples of Klein demonstrate a great deal of effort by an extremely talented and dedicated mental calculator, far beyond reasonable expectations for the rest of us. But modern calculators have eclipsed Klein, claiming records of mere seconds for particular attempts. One such calculator is Gert Mittring of Germany. On August 25, 2011, Mittring won his eighth consecutive title in the Mental Calculations World Championship, evidence of formidable talent. In 2004 Mittring (shown here with a different record attempt) achieved a record of 11.8 seconds for the 13th root of

\[ 70664373816742861022340088302401573757042331707026 \\
32731269721516000395709065419973141914549389684111 \]

Writing in *Spiegel Online* [Mittring 2004], Mittring describes one strategy for finding this particular root, as translated below:

There are different strategies in the present problem of finding the solution. I will try to explain a variation that I’ve used. As you like, you may devise alternative strategies. It is desirable in any case to have a “memory-economic” variant. The determination of the 8-digit (integer) 13th root of a 100-digit number is done in three major steps:

1. The estimate of the logarithm.
2. The division by 13 (the root exponent). The result is the logarithm of the solution.
3. The conversion of the logarithm to its antilogarithm.

It is sufficient to look at the first six digits of the 100-digit number. Rounded, it starts with \( 706644 \times 10^{94} \). How can the logarithm be estimated effectively? I only know the logarithms of the primes up to 100 (2, 3, 5,…, 97). That amounts to 25 logarithm values to 7 decimal places each, so the memory effort is equivalent to 15 phone numbers.

**The first big step:** A first estimate is the following simple application of the rules of logarithms by factoring the number \( 706644 \times 10^{94} \) into prime factors and then summing them up (where
\[
\log 2 + \log 5 = 1);
\]
\[
\log(706644 \times 10^{94}) \approx 94 + 2\log 2 + 5\log 3 + \frac{6\log 3 + 2\log 5 + \log 29}{2}
\]
\[
= 95 + 2 + 8\log 3 + \frac{\log 29}{2}
\]
\[
\approx 99.849199 \ldots \text{ (the exact value would be 99.849200\ldots)}
\]

The second step is simple: Through my experience with multiplication I immediately know that 7.68 \times 13 = 99.84. Right away I get the additional decimal places 07077 (rounded up, and knowing very well that the estimate for the original logarithm is a lower bound).

The third step: In the last step I have to find the antilogarithm of 7.6807077. A rough linear approximation over
\[
\log 48 = 4\log 2 + \log 3 \quad \text{and} \quad \log 47
\]
gives me the first digits of the solution: 4794. It is apparent that
\[
\log 4794 \times 10^4 = 4 + \log 2 + \log 3 + \log 17 + \log 47 \quad \text{(where log 47 is already known above)}
\]
\[
\approx 7.6806980
\]

The difference now is only 97 \times 10^{-7}. Obviously, 97 times 'a bit over 11 units' still needs to be added. The estimated solution is then 'a little over' 47941067.

As a check, one can analyze the end digits of the original number. Because the 100-digit number ends in 11, a rule tells me that the solution needs to end in 71. Therefore, everything speaks in favor of 47941071, which I then spoke out and which was indeed the right answer.

Let’s study Mittring’s third step for finding the antilogarithm of 7.6807077. For \( N = 10^{7.6807077} \), he uses his knowledge of logarithms to find \( N \) such that \( \log N = 7.6807077 \):

a) Mittring would know that \( \log 47 = 1.6721 \). The original number minus 6 yields 1.6807077 which is greater than \( \log 47 \) by 0.0086. So he checks \( \log 48 \), and in this case he factors it into primes \( 2^4 \times 3 \) and finds \( \log 48 \) as \( 4\log 2 + \log 3 = 1.6812 \), which is a little high by 0.0005. In practice, Mittring has no doubt memorized the table of logarithms on the right that spans the range of the initial two digits of the 13th root.

b) The “rough linear approximation” is a linear interpolation between \( \log 47 \) and \( \log 48 \). The difference between the values of \( \log 48 \) and \( \log 47 \) is 0.0091 per the table, and \( \log 48 \) is 0.0005 too high, so

\[
\frac{0.0005}{0.0091} = 0.06 \text{ is subtracted from 48 to yield 47.94. Adding the 6 that was subtracted from the original number of 7.6807077 gives 6 + log 47.94 or log 10^6 + log 47.94 or log 47940000. Now 4794000 becomes our initial estimate of the root.}
\]
c) Now Mittring finds the exact value of $\log_{10}47940000$ in order to find its difference from the exact value of 7.6807077. From his innate knowledge of numbers, he factors 47940000 into $10^4 \times 2 \times 3 \times 17 \times 47$, so $\log_{10}47940000 = 4 + \log_2 3 + \log_{17} + \log_{47}$, which he adds up from memorized values of logarithms to get $\log_{10}47940000 = 7.6806980).

d) Now $\log_{10}47940000$ is 0.0000097 lower than 7.6807077. How much would we increase 47940000 if we increase its log by 97? In other words, for $n = 47940000$ what is $\Delta n$ for a given increase of $\Delta (\log n)$? Mittring at least implicitly uses the linear approximation

$$\Delta n \approx 2.3 n \Delta (\log n)$$

$$= 2.3 \times 4.794 \times 10^7 \times 97 \times 10^{-7} \approx 11 \times 97 = 1067$$

and so 47941067 is our next estimate. The “bit over 11 units” that is multiplied by 97 in Mittring’s explanation is evidently from approximating $2.3 \times 4.8$.

e) The final two digits of a 13th root of a power ending in 1, 3, 7, or 9 are easily found from the final two digits of the power (the other endings of 13th powers do not provide unique endings of their roots). The table on the right shows the two digit endings of 13th roots for any two-digit endings of the powers. For any final two digits of the power shown in the body of the table, the corresponding digits of the root are found at the edges. The last two digits of the power are 11, so the root ends in 71. The estimate of 47941067 is therefore modified to 47941071.

Did Gert Mittring do all this in 11.8 seconds? He described it as simply one strategy he has used, a variant that is “memory-economic,” but it requires significant knowledge of factors and logarithmic values. The details of the presentation of the problem and the timing parameters of the test are not known, but it seems inconceivable that this was the method used to achieve this record time. Perhaps this attempt was achieved through a different, “memory-intensive” method. In the next section we will discuss another mental calculator who bested Mittring’s time and refers to these as simply tests of memory, albeit of prodigious memory.

**Alexis Lemaire**

Less than a month after Gert Mittring’s 2004 record of 11.8 seconds, Alexis Lemaire of France is reported to have extracted the 13th root of the number shown below in 3.625 seconds. As presumably the case with Mittring, this time included the time to write the answer of 45792573:
What are we to make of this feat? On Lemaire’s website [Lemaire], he relates how he has moved on to extracting 13th roots of 200-digit numbers because the former “can now only be a record of memorization.”

[The 3.625 record time] means the 13th root of a 100-digit number is an immediate calculation (1 second), and the recordholder will be the one with the fastest time for writing, not with the fastest time for calculating. Every left part and right part of the 13th root of a 100-digit number can be memorized: there are only 3 and 4 digits to be memorized. Therefore the 13th root of a 100-digit number can be only a record of memorization, whereas the 13th root of a 200-digit number is a true task of mental calculation.

Lemaire here downplays the difficulty of memorizing all the starting and ending combinations; this is a stunning achievement of memorization. Referring to this record time and the number of possible answers, Lemaire in another paper [Lemaire 2009] states

Nearly 8 million combinations have been learned beforehand (consciousness of the future) through a generalization axiom which compresses these numbers into softer rules by working out patterns. … Furthermore, this reverse artificial intelligence uses fuzzy sets to compute faster when dealing with the central most difficult part of the calculation; the fuzzier the computation, the faster it is but also the less accurate. We use this point to break records set after a great number of attempts.

The complete set does not need to be memorized; a “fuzzy” but fast attempt is made and if the answer is incorrect the next number is attempted. A competitor may also be asked a familiar number by chance. It is important to remember that mental calculators make many, many attempts at record times. Lemaire remarks on his website that it took 742 attempts in 2005 to beat his previous record in the 200-digit realm. This is not a criticism—athletes do the same—but simply an aspect of the competition.

The Current State of 13th Root Extraction

Wim Klein once said, “What is the use of extracting the 13th root of 100 digits? ’Must be a bloody idiot,’ you say. No. It puts you in the Guinness Book, of course” [Smith, 1983]. But in fact Guinness World Records dropped the category of 13th roots of 100-digit roots because the result is so dependent on the particular value of the power. And with the memorization capabilities witnessed today (most visible in the memorization of \( \pi \)), the field does not have the intellectual depth it once enjoyed.
Today a set of 13th roots is required for the world record maintained by a separate organization, Rekord-Klub SAXONIA, and this set purposely includes even-numbered final digits of the powers that allow multiple combinations of final digits [Rekord-Klub SAXONIA]. It appears that no one has attempted this, and in fact it is likely that only Wim Klein, who passed away in 1986, could do it.

**The Doerfler-Forster Method**

Can it be possible for us to extract 13th roots of 100-digit numbers without devoting portions of our lives to it? With a basic talent in mental arithmetic and some study, it can certainly be done, even if not in record time. We present a method that involves no logarithms, no antilogarithms and no factoring, one that works with 13th powers that end in 1, 3, 7 or 9 (the cases attempted by record holders). The memorization consists of one table and a few formulas. Performing this calculation with pen and paper and no memorization is perhaps the best way to become comfortable with the process.

The procedure as written here seems more complicated than it is in use. In practice insignificant digits are dropped as they occur, and hundreds digits are dropped when finding a result mod 100.

**The Last Four Digits of the Root**

First, we find the last 4 digits of the root. We find these from the last 4 digits of the power, which we denote as \(dcba\) where \(a\) is the final digit.

1. The last digit of the root is the same as that of the power.
2. The second-to-last digit of the root (as evident in the earlier table) is:
   \[
   7b \mod 10 \quad \text{for } a = 1 \text{ or } 7 \\
   7(b - 2) \mod 10 \quad \text{for } a = 3 \text{ or } 9
   \]
3. The next pair of digits of the root (the 3rd and 4th from the end) are found from the formulas below. The mod 100 operation at the end is the remainder when divided by 100, so it is the last two digits if the result is positive, or 100 minus the last two digits if the result is negative. In fact, the formulas are much simpler than they appear because only the last two digits are kept in each term. The ceiling function \(\lceil \cdot \rceil\) rounds the result up to the next integer, and the floor function \(\lfloor \cdot \rfloor\) rounds the result down to the previous integer.
   \[
   70d - 23c + 26b^2 + b(20c + 8) - \lceil b/3 \rceil \mod 100 \quad \text{for } a = 1
   
   70d + 17(c + 1) + 32b^2 + b(40c + 42) - \lfloor b/3 \rfloor \mod 100 \quad \text{for } a = 3
   \]

For \(a = 7\), we find \(10000 - dcba\), then find the last four digits of the root from this
new value that ends in 3, then subtract the answer from 10000

For $a = 9$, we find $10000 - dcba$, then find the last four digits of the root from this new value that ends in 1, then subtract the answer from 10000

A fast way to subtract $dcba$ from 10000 is to subtract each digit from 9 and add 1 to the result.

Now we have the last four digits of the root and we can proceed to find the first four digits.

The First Four Digits of the Root

It remains to find the first 4 digits of the root from the initial 5 digits of the power. We can do this by memorizing a short table of root vs. power starting digits, and then applying an offset.

The table to memorize is shown on the right. These values represent a minimum distribution that provides 4-digit accuracy over the interval of 100-digit powers. The values are actually accurate to 5 digits rather the 4 digits, so in fact the next digit is 0 for each value. Therefore, we have values for 5-digit accuracy that only require memorizing 4-digit numbers. Also, the values of $n$ are convenient multipliers to use in our formula.

The steps to find the first 4 digits of the root are:

1. Find $S$ as the first 5 (rounded) digits of the power divided by 10, so the 5th digit follows the decimal point. Find the values of $P$ in the table on either side of $S$. If $S$ is less than $1/3$ the distance between these values of $P$, choose the $R$ and $P$ from the lower entry row, otherwise choose $R$ and $P$ from the higher row.

2. Find the difference $D = S - P/10000$ to the 4th decimal place. Then find the correction below to 3 decimal places and add it to $R$:

$$ \text{correction} = nD - \frac{6(nD)^2}{R} $$

Generally only the first term is required, but if $D$ is larger the second term may provide an additional small correction that can be taken to a digit or two. Note that $R$ ranges from 42 to 49 so we can replace $6/R$ with $1/7$ or $1/8$ depending on which end of the range $R$ lies on.

3. Then merge the first 4 digits with the last 4 digits to get the most reasonable last digit of the first four digits. For example, assume the first four digits were 2222. If the last four were 3333 the answer would be a simple merge to 22223333, but if the last four were 7777 the last digit 2 in 2222 would be adjusted down to 1 to get 22217777 because 1.7 is closer to 2.0 than 2.7.
If the first digit of the last four digits is 5, we would use the 5th digit we calculated for the first digits to decide what is the closest merge.

**Example Problems**

The exercises presented below are intended to show the method in use with published problems faced by mental calculators. As mentioned earlier, achieving the record times listed here would require a great deal more memorization and practice.

**Example 1: Alexis Lemaire Record 13th Root Problem (13.55 seconds)**

The first example finds the 13th root of the following number:

\[
2928811583487520106055367352783652122196502020937
\]
\[
1392842551086152669633464222587770308279739304053
\]

The last digits of the power are 4053, so the last digit of the root is 3, and 

\[7(5 - 2) \mod 10 = 1\]

so the last two digits of the root are 13. Then we work left-to-right from the formula below, ignoring any hundreds digits as we go:

\[
70(4) + 17(0 + 1) + 32(5)^2 + 5(40(0) + 42) - \lfloor \frac{5}{3} \rfloor \mod 100 = 06
\]

so the last 4 digits of the root are 0613.

The first 5 digits of the power (rounded) are 29288 which we divide by 10 to get 2928.8, and the closest match in the table is 2869 so our first approximation is 44.73. Now 

\[D = 2928.8 - 2869/10000 = 0.0060\]

and the first correction 

\[nD = 12(0.0060) = 0.072\]

which we add to 44.73 to get a closer value of 44.802. The second term 

\[0.0060^2/7\]

is too small to make a difference. Therefore we merge 44.80 and 0613, and we end up with 44800613.

**Example 2: Gert Mittring’s Record 13th Root Problem (11.8 seconds)**

This example finds the 13th root of a number ending in 1:

\[
70664373816742861022340088302401573757042331707026
\]
\[
32731269721516000395709065419973141914549389684111
\]

The last digits of the power are 4111, so the last digit of the root is 1, and 

\[7b \mod 10 = 7\]

so the last two digits of the root are 71. Then we work left-to-right from the formula below, ignoring any hundreds digits as we go:

\[
70(4) - 23(1) + 26(1)^2 + (1)(20(1) + 8) - \lfloor \frac{1}{3} \rfloor \mod 100 = 10
\]
so the last 4 digits of the root are 1071.

The first 5 digits of the power (rounded) are 70664 which we divide by 10 to get 7066.4, and the closest match in the table is 7398 so our first approximation is 48.11. Now $D = \frac{7066.4 - 7398}{10000} = -0.03316$ and the first correction $nD = 5(-0.03316) = -0.166$ which we add to 48.11 to get a closer value of 47.944. Now $-0.166^2/8$ is very small, about 0.003, and subtracting this gets 47.941. Therefore we merge 4794 and 1071, and we end up with 47941071.

**Example 3: Record 13th Root Problem by Wim Klein (1 minute 28 seconds) and Gert Mittring (39 seconds)**

Here is an example for a power ending in 7:

88008443440489299575219015772236417859411720052615
65487280650870412023307854274990144578442271602817

The last digit is a 7, so we replace the final digits 2817 with $10000 - 2817 = 7183$. The last digit of this root is 3, and $7(8 - 2) \mod 10 = 2$ so the last two digits of the root are 23. Then we work left-to-right from the formula below for 7183, ignoring any hundreds digits as we go:

$$70(7) + 17(1+1) + 32(8)^2 + 8(40(1) + 42) - \left\lfloor \frac{8}{3} \right\rfloor \mod 100 = 26$$

so the last 4 digits of the root for 7183 are 2623. Then we find $10000 - 2623 = 7377$ as the last four digits of our original power ending in 2817.

The first 5 digits of the power (rounded) are 88008 which we divide by 10 to get 8800.8, and the closest match in the table is 9437 so our first approximation is 49.02. Now $D = \frac{8800.8 - 9437}{10000} = -0.0636$ and the first correction $nD = 4(-0.0636) = -0.254$ which we add to 49.02 to get a closer value of 48.766. The second term $-0.25^2/8 = 0.008$ so we subtract that to get 48.758. Therefore we merge 48.758 and 7377, and the closest 8-digit match is 48757377.

**Additional Examples**

As an additional exercise, you may want to try this record by Alexis Lemaire (3.625 seconds):

38934589793526802773496632556519305532657006082154
4981718856605442717204610395223260479910745354353

The answer is given at the very end of this paper.

Another exercise is this problem of Gert Mittring (11.8 seconds):

34288725041442601391808603643426837852427296517260
61936285121642529526002848517356482932010681285881
This answer can be found in the earlier photograph of Gert Mittring.

Additional examples can be created by installing the Python compiler from

http://www.python.org/download/

Then download and save this Python script:


Double-clicking on the file will then launch a window that provides 13th powers ending in 1, 3, 7 and 9 along with their roots.

Conquering the 13th Root

Despite the perception of a general decline in mathematical ability, there are a number of outstanding mental calculators today who use advanced algorithms and memory techniques to eclipse their forerunners. The 13th root problem is clear evidence of this advancement. The early attempts by lightning calculators to find 13th roots of 100-digit numbers were beyond ordinary comprehension at the time, and the practitioners were considered marvels of nature. Today this problem is performed an order of magnitude faster and sometimes, with enough attempts, in a matter of seconds.

However, these feats require an astonishing dedication and number knowledge that still places these lightning calculators in a category beyond the understanding of the vast majority of people. We have attempted here to present typical approaches taken by lightning calculators to this problem, both past and present, and we have described a new method that allows the mathematically-inclined person—with a reasonable degree of time and effort—to extract 13th roots of 100-digit numbers, and to do so at speeds that were once considered unthinkable.
Appendix: Derivation of the Doerfler-Forster Method

The final digit of the root is always the same as the final digit of the power since the power is of the form $4k + 1$. The formula for the second-to-last digit is apparent from the table of 2-digit endings presented above. The formulas for the two digits prior to these were found by manually searching for patterns in 4-digit tables of 13th roots, a task too mind-numbing to relate. Lemaire described this in an earlier excerpt quoted in this paper as “working out patterns.” The formulas were then verified over all the 4-digit endings by a software program.

The Newton-Raphson iterative approximation for solving the equation $a^p = N$ is given by

$$a_{n+1} = a_n + a_n \frac{N - a_n^p}{pa_n^{p-1}}$$

or

$$correction = a_{n+1} - a_n = a_n \frac{N - a_n^p}{pa_n^{p-1}}$$

For an initial estimate $R$ of $N^{1/13}$, we can improve this estimate with the correction from one iteration:

$$correction_1 = RN - R^\frac{13}{13}$$

Defining $n = R^{13} = 13R^{12}$ and $D = N - R^{13}$, the correction we add to the initial estimate is

$$correction_1 = nD$$

Rather than attempt a second iteration of the Newton-Raphson method, we can simply add another correction called the Chebyshev term [Doerfler 1993].

$$correction_2 = \frac{p - 1}{2p^2a_n^{p-1}} \left( \frac{N - a_n^p}{a_n^{p-1}} \right)^2 = \frac{12}{2 \times 13^2 \times R} \left( \frac{N - R^{13}}{R^{12}} \right)^2 = \frac{6}{R} \left( \frac{N - R^{13}}{13R^{12}} \right)^2 = \frac{6(nD)^2}{R}$$

The total correction is the sum of $correction_1$ and $correction_2$:

$$correction = nD - \frac{6(nD)^2}{R}$$
A table was created in software of all numbers \( n \) to 4 decimal places between \( 4 \times 10^{-22} \) and \( 32 \times 10^{-22} \). A column was added for \( R \) from the equation \( n = \frac{1}{13} R^{12} \), and another column was added for \( P \), where \( P = R^{13} \times 10^{-18} \). This table was manually searched for rows in which \( n \) (without the exponent) was an integer to high accuracy, while \( R \) and \( P \) rounded to 4 digits with 5-digit accuracy. These were candidates for rows in the memorized table, where the powers of 10 are incorporated into the method rather than the table. Finally, the values were tested until there was a sufficient distribution of table rows to provide accuracy over the range of possible roots \( R \) to 3 decimal places in the correction formula. The result is a table that provides sufficient accuracy of results while minimizing the task of memorization.

REFERENCES


Lemaire, A. and Rousseaux, F. 2009. *Hypercalculation for the mind emulation*, AI and Soc, 24:191-196. Available from SpringerLink at [http://www.springerlink.com/content/c3v31q6k0r4854n3/](http://www.springerlink.com/content/c3v31q6k0r4854n3/)


**Answer to Additional Example #1:**

\[
\sqrt[13]{38934589793526802773496632556519305532657006608215449817188566054427172046103952232604799107453543533} = 45792573
\]